

## PROBABILISTIC ANALYSIS OF THE $K$ -SERVER PROBLEM ON THE CIRCLE

ARIS ANAGNOSTOPOULOS,\* *Sapienza University of Rome*

CLÉMENT DOMBRY,\*\* *Université de Poitiers*

NADINE GUILLOTIN-PLANTARD,\*\* *Université de Lyon*

IOANNIS KONTOYIANNIS,\*\*\*\* *Athens University of Economics & Business*

ELI UPFAL,\*\*\*\*\* *Brown University*

### Abstract

We consider a stochastic version of the  $k$ -server problem, in which  $k$  servers move on a circle to satisfy randomly generated requests. The requests are independent and identically distributed, according to an arbitrary distribution that is either discrete or continuous. The cost of serving a request is the distance that a server needs to travel in order to reach the request. The goal is to minimize the steady-state expected cost induced by the sequence of requests. We study the performance of a greedy strategy focusing, in particular, on its convergence properties and the interplay between the discrete and continuous versions of the process. Finally, we show that in the case of  $k = 2$  servers the greedy policy is optimal.

*Keywords:*  $k$ -server; probabilistic analysis; on-line algorithms

2000 Mathematics Subject Classification: Primary 60J05

Secondary 68W40

---

\* Postal address: Department of Informatics and System Sciences, Sapienza University of Rome, Via Ariosto 25, 00185 Rome, Italy. aris@dis.uniroma1.it.

\*\* Postal address: Laboratoire de Mathématiques et Applications, CNRS UMR 6086, Université de Poitiers, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France. clement.dombry@math.univ-poitiers.fr.

\*\*\* Postal address: Université de Lyon ; Université Lyon 1 ; INSA de Lyon, F-69621 ; Ecole Centrale de Lyon ; CNRS, UMR5208, Institut Camille Jordan, 43 bld du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France, e-mail: nadine.guillotini@univ-lyon1.fr.

\*\*\*\* Postal address: Department of Informatics, Athens University of Economics & Business, Patission 76, Athens 10434, Greece. yiannis@aueb.gr.

\*\*\*\*\* Postal address: Computer Science Department, Brown University, Providence RI 02906, US. eli@cs.brown.edu.

## 1. Introduction

The *k-server problem* first posed by Manasse, McGeoch, and Sleator [6, 7] is central to the study of on-line algorithms. Let  $k \geq 2$  be an integer and consider a metric space  $\mathbb{M} = (M, d)$  where  $|M| > k$  is a set of points and  $d$  is the metric over  $M$ . A *k-server problem* is defined by a sequence of *requests*,  $r_1, r_2, \dots$ , where each  $r_i \in M$ . A *k-server algorithm* controls  $k$  *servers* that reside in, and move between, points of  $M$ . A request  $r_i$  is satisfied when the algorithm moves a server to that point. The algorithm must satisfy request  $i$  before it sees any future request. The cost of serving a sequence of requests is the sum of the distances that servers have to traverse in order to satisfy these requests. The *k-server problem* models a number of important on-line computation problems such as paging and disk-scheduling [2]. So far, most studies of the *k-server problem* have focused on the *competitive analysis* of the problem, culminating in Koutsoupias and Papadimitriou's [5] seminal results, establishing a worst-case competitive ratio of  $\Theta(k)$  for the problem.

An alternative approach to the study of on-line algorithms is *probabilistic analysis*. Instead of taking a worst-case approach, we assume that the input (in this case, the sequence of requests) follows some distribution, and we analyze the expected cost of a given algorithm over this input distribution. As in the worst-case scenario, the algorithm satisfies each request without information about the sequence of future requests, and we are interested in algorithms that are not tuned to a particular input distribution.

### 1.1. The Stochastic Model

In this work we study a stochastic version of the *k-server problem* in which the  $k$  mobile servers move on the circle of circumference  $k$ , which we denote by  $\mathbb{T}$  and identify with the interval  $[0, k)$ . Throughout, we focus on the long-term behavior of the process induced by the following *greedy* algorithm: "at each step  $i$ , send the server that is closest to the request  $r_i$ ." We analyze and compare two different models; in the continuous model, requests arrive anywhere on the circle  $\mathbb{T} = [0, k)$  according to some distribution with a continuous density; in the discrete case, requests may arrive only at  $\ell$  discrete stations, where  $k < \ell < \infty$ , and the positions of the stations are in  $\mathbb{T}_\ell = \{ik/\ell; 0 \leq i \leq \ell - 1\} \subset \mathbb{T}$ . The circle  $\mathbb{T}$  is endowed with the distance,

$$d_0(x, y) = \min\{|x - y|, |x - y - k|, |x - y + k|\},$$

and  $\mathbb{T}_\ell$  with the induced distance.

We assume that the requests are independent and identically distributed (i.i.d.) according to a distribution supported either on  $\mathbb{T}$  (in the continuous case) or on  $\mathbb{T}_\ell$  (in the discrete case); their common distribution is denoted either by  $\mu$  or  $\mu_\ell$ , respectively. For the sake of simplicity, we will always assume that  $\mu$  has a strictly positive density with respect to Lebesgue measure on  $\mathbb{T}$  and that  $\mu_\ell$  has positive mass on all points  $x \in \mathbb{T}_\ell$ .

In the continuous setting, we can model the process as a Markov chain  $(S_n)_{n \geq 0}$  evolving on the state

space,

$$\Sigma = \{\mathbf{p} = (p_0, p_1, \dots, p_{k-1}) \in \mathbb{T}^k; p_j \leq p_{j+1} \leq \dots \leq p_{k-1} \leq p_0 \leq p_1 \leq \dots \leq p_{j-1}, \text{ for some } j\}.$$

Here  $p_i$  corresponds to the position of the  $i$ th server. In the discrete setting, the servers move only within the  $\ell$  stations, and accordingly the state space is  $\Sigma_\ell = \Sigma \cap \mathbb{T}_\ell^k$ . To avoid ambiguity, in the case when the two closest servers to a given request  $r_i$  are equidistant from  $r_i$ , we specify that the greedy algorithm moves the server which is to the left of the request (where the circle is oriented in the usual way). In addition, if two or more servers are on the same location (i.e., we have  $p_i = p_{i+1} = \dots = p_j = x$  for  $i < j$ , or  $p_i = p_{i+1} = \dots = p_{k-1} = p_0 = \dots = p_j = x$  for  $i > j$ ), and the new request is closer to  $x$  than to any other server, we choose server  $j$  if the request is on the left of  $x$  and server  $i$  if it is on the right of  $x$ . Note that, under these assumptions, the relative order of the servers on the circle does not change, and that, if at some point all the servers are on different locations, then they will never be on the same location in the future of the process.

The performance of the greedy algorithm is measured in terms of the total distance travelled by the servers to fulfill the requests. Let  $d$  be the distance on  $\mathbb{T}^k$  defined by

$$d(\mathbf{p}, \mathbf{p}') = \sum_{i=0}^{k-1} d_0(p_i, p'_i).$$

Recalling that  $S_n$  denotes the position of the servers after satisfying the  $n$ th request, the *cost* of the  $n$ th request is given by  $C_n = d(S_{n-1}, S_n)$ . We are interested in the asymptotic behavior of *the average cost over a run of  $N$  requests*, namely,

$$\frac{1}{N} \sum_{n=1}^N C_n = \frac{1}{N} \sum_{n=1}^N d(S_{n-1}, S_n).$$

## 2. Results

### 2.1. Convergence to a Stationary Distribution

In the discrete setting, since requests may arrive at all stations with positive probability, the finite state space Markov chain  $(S_n)_{n \geq 1}$  has the following structure; let,

$$\tilde{\Sigma} = \{\mathbf{p} = (p_0, p_1, \dots, p_{k-1}) \in \mathbb{T}^k; p_j < p_{j+1} < \dots < p_{k-1} < p_0 < p_1 < \dots < p_{j-1}, \text{ for some } j\},$$

and  $\tilde{\Sigma}_\ell = \tilde{\Sigma} \cap \mathbb{T}_\ell^k$ ; then every state in  $\Sigma_\ell \setminus \tilde{\Sigma}_\ell$  is transient and  $\tilde{\Sigma}_\ell$  is the unique closed communication class and it is aperiodic. As a consequence (see, for example, [9]), the Markov chain  $(S_n)$  has a unique stationary distribution, say  $\pi_\ell$ , supported on  $\tilde{\Sigma}_\ell$ , the distribution of the chain  $(S_n)$  converges to  $\pi_\ell$  exponentially fast, and the law of large numbers holds. In particular, the average cost over  $N$  steps converges almost surely to the mean cost under  $\pi_\ell$ , as  $N \rightarrow \infty$ .

The main contribution of this work is to obtain similar results in the continuous case. Under the assumption that  $\mu$  has a positive density, we show in Theorem 1 that the process has a unique stationary distribution  $\pi$ , to which it converges exponentially fast.

**Theorem 1.** *Assume that  $\mu$  has a density with respect to the Lebesgue measure on  $\mathbb{T} = [0, k)$  that is uniformly bounded away from 0. Then the Markov chain  $(S_n)_{n \geq 0}$  on  $\Sigma$  induced by the greedy algorithm has a unique stationary distribution  $\pi$ , and:*

- for all  $n \geq 1$ ,  $\sup_{A, s_0} |P(S_n \in A | S_0 = s_0) - \pi(A)| \leq c_1 e^{-c_2 n}$ , the supremum being taken over all measurable sets  $A$  of  $\Sigma$  and all  $s_0 \in \Sigma$ ;
- $\frac{1}{N} \sum_{n=1}^N C_n \rightarrow \mathbb{E}_\pi(C)$  with probability 1, as  $N \rightarrow \infty$ ;
- $\Pr\left(\frac{1}{N} \sum_{n=1}^N C_n > \mathbb{E}_\pi(C) + \epsilon\right) \leq c_3 e^{-c_4(\epsilon)n}$  for all  $\epsilon > 0$ ,  $n \geq 1$ ,

where the finite constants  $c_1, c_2, c_3, c_4(\epsilon)$  may depend on  $k$  but not on  $n$ ;  $c_2 > 0$  and  $c_4 = c_4(\epsilon)$  is strictly positive for all  $\epsilon > 0$ ;  $C_i = d(S_{i-1}, S_i)$  is the cost of the  $i$ th request; and  $\mathbb{E}_\pi(C)$  is the expected value of the one-step cost for a stationary version of the chain.

In view of the well-known convergence results in [8] and the more recent large deviations estimates in [4], in order to prove the theorem it suffices to verify that the Markov chain fulfills Doeblin's condition; see [8, page 391]. The proof is given in Appendix A.

In addition to the results of the theorem, since the cost function is bounded by definition, the Doeblin (or uniform ergodicity) property implies finer asymptotic results for the convergence of the averaged cost  $\frac{1}{n} \sum_{i=1}^n C_i$ . In particular, it is asymptotically normal, and the convergence to normality can be further refined via an Edgeworth expansion. Moreover, the exponential bound on the excess-cost probability can be sharpened to a precise asymptotic formula along the lines of the Bahadur-Rao refinement to the classical large deviations principle [3]. See [4] for details.

## 2.2. Convergence of the Discrete Model to the Continuous One

When the number of stations  $\ell$  in the discrete model is large, we naturally expect that it should be possible to approximate the continuous model by an appropriately defined version of the discrete one. A precise asymptotic version of the above statement is established in the following proposition.

**Proposition 1.** *Suppose that  $\{\mu_\ell\}$  is a sequence of request distributions on  $\mathbb{T}_\ell$ ,  $\ell \geq 1$ , such that  $\mu_\ell \rightarrow \mu$  weakly, as  $\ell$  goes to infinity, for some continuous request distribution  $\mu$  on  $\mathbb{T}$ . Then, the corresponding sequence  $\{\pi_\ell\}$  of stationary distributions for the discrete chains induced by each of the  $\mu_\ell$  also satisfies  $\pi_\ell \rightarrow \pi$  weakly, as  $\ell \rightarrow \infty$ , where  $\pi$  is the stationary distribution of the continuous chain induced by  $\mu$ .*

*Proof.* If the servers are in position  $\mathbf{p} \in \Sigma$  and a request arrives at site  $x \in \mathbb{T}$ , then the new position of the servers after service is

$$Q(\mathbf{p}, x) = (p_0, \dots, p_{i-1}, x, p_{i+1}, \dots, p_{k-1}) \quad (2.1)$$

where  $p_i$  is the nearest point of  $x$  (and the left such point in case of equality). The continuity set of the map  $Q$  is exactly the set of points  $(\mathbf{p}, x)$  such that there is a unique point  $p_i$  nearest to  $x$ .

Now suppose the sequences  $\{\mu_\ell\}$  and  $\{\pi_\ell\}$  are as in the statement of the proposition. According to Theorem 2 below, the stationary measures  $\pi_\ell$  converge weakly to  $\pi$ , and so do the corresponding expected costs. Note that the assumptions of the theorem are always satisfied when  $\mu$  is diffuse and, in particular, when  $\mu$  has a density.  $\square$

**Theorem 2.** *Let  $C_Q \subset \Sigma \times \mathbb{T}$  denote the continuity set of the map  $Q$  defined in equation (2.1). Assume that the Markov chain  $(S_n)_{n \geq 0}$  associated with the request distribution  $\mu$  has a unique invariant probability measure  $\pi$ , and that, for every probability measure  $\pi'$  on  $\mathbb{T}^k$ ,*

$$(\pi' \times \mu)(C_Q) = 1.$$

*Then  $\pi_\ell \rightarrow \pi$  weakly and  $E_{\pi_\ell}(C) \rightarrow E_\pi(C)$ , as  $\ell \rightarrow \infty$ , where  $E_{\pi_\ell}(C)$  and  $E_\pi(C)$ , denote the expected costs under the stationary versions of the corresponding chains, respectively.*

*Proof.* The only point of possible concern is that the map  $(\mathbf{p}, x) \mapsto Q(\mathbf{p}, x)$  may not be continuous everywhere, unless the minimum distance between  $x$  and  $\mathbf{p}$  is attained at a single point.

The invariance of  $\pi_\ell$  implies that  $Q$  maps the measure  $\pi_\ell \times \mu_\ell$  to  $\pi_\ell$ . Moreover, the state space  $\Sigma$  is compact, and hence the sequence  $(\pi_\ell)$  is tight. Let  $\pi'$  be an accumulation point of  $(\pi_\ell)$  and let  $(\pi_{\ell_i})_{i \geq 0}$  be a subsequence of  $(\pi_\ell)$  converging to  $\pi'$ . By taking the limit  $i \rightarrow \infty$  in  $Q(\pi_{\ell_i} \times \mu_{\ell_i}) = \pi_{\ell_i}$ , we obtain that  $Q(\pi' \times \mu) = \pi'$ , that is,  $\pi'$  is invariant for  $(S_n)$ ; the hypothesis  $(\pi' \times \mu)(C_Q) = 1$  is used here to apply the continuous mapping theorem [1]. By the uniqueness of the invariant measure (Theorem 1) we have that  $\pi' = \pi$ . Since we have a unique accumulation point, the convergence of the sequence  $(\pi_\ell)$  follows, completing the proof.  $\square$

We provide further insight into the correspondence between the discrete and the continuous models by constructing a *coupling* between the corresponding chains. Consider the continuous model with a sequence of requests  $(r_n)_{n \geq 1}$  that are i.i.d. with continuous distribution  $\mu$  on  $\mathbb{T}$ . Let  $(S_n)_{n \geq 0}$  denote the associated Markov chain on  $\Sigma$  and let  $\Theta_\ell : \mathbb{T} \rightarrow \mathbb{T}_\ell$  be the discretization map defined by,

$$\Theta_\ell(x) = \frac{k}{\ell} \left\lfloor \frac{\ell}{k} x \right\rfloor.$$

Similarly, let  $(r_n^\ell)_{n \geq 1} = (\Theta_\ell(r_n))_{n \geq 1}$  denote the corresponding discretized requests, so that the sequence  $(r_n^\ell)$  is also i.i.d., with discrete distribution  $\mu_\ell$  given by  $\mu_\ell(x) = \mu(\lfloor x, x + k/\ell \rfloor)$  for  $x \in \mathbb{T}_\ell$ . We consider the Markov chain  $(S_n^\ell)$  defined by the initial state  $S_0^\ell = \Theta_\ell(S_0)$  and the sequence of requests  $(r_n^\ell)_{n \geq 1}$ . Note that

$S_n^\ell$  is in general *not* the same as  $\Theta_\ell(S_n)$ . We assume that  $\ell$  is large enough so the image under  $\Theta_\ell$  of the initial positions of the  $k$  servers is a set of  $k$  distinct points.

As the number of stations  $\ell$  goes to infinity,  $\Theta_\ell$  converges to the identity map and  $\mu_\ell$  converges weakly to  $\mu$ , so we naturally expect that the Markov chains  $(S_n^\ell)$  and  $(\Theta_\ell(S_n))$  will also be “close” for large  $\ell$ . In order to quantify this closeness, we examine the *(de-)coupling time*,

$$T_\ell = \inf\{n \geq 1; S_n^\ell \neq \Theta_\ell(S_n)\}.$$

Our next result is the following:

**Proposition 2.** *Let  $\delta(\epsilon) = \sup\{\mu([x, x + \epsilon]) ; x \in \mathbb{T}\}$  for  $\epsilon > 0$ , and assume that  $\delta(k/\ell)$  is strictly positive. Then the distribution of  $T_\ell$  stochastically dominates a geometric distribution with parameter  $k\delta(k/\ell)$ , i.e.,*

$$\Pr(T_\ell > n) \geq \left(1 - k\delta\left(\frac{k}{\ell}\right)\right)^n.$$

As a consequence,  $\mathbb{E}(T_\ell) \geq (k\delta(k/\ell))^{-1}$ . Moreover, if density of  $\mu$  is uniformly bounded above by some constant  $c' > 0$ , then  $\delta(\epsilon) \leq c'\epsilon$  for all  $\epsilon > 0$ , and  $\mathbb{E}(T_\ell) \geq \ell/(c'k^2)$ .

*Proof.* The stochastic domination is a simple consequence of the following estimate,

$$\Pr(S_{n+1}^\ell = \Theta_\ell(S_{n+1}) | S_n^\ell = \Theta_\ell(S_n)) \geq 1 - k\delta\left(\frac{k}{\ell}\right),$$

which we show next. Assume that at time  $n$  we have  $S_n^\ell = \Theta_\ell(S_n)$ . For  $0 \leq i \leq k-1$ , let  $q_n^\ell(i)$  be the midpoint between the  $i$ th and the  $(i+1)$ st server in the discrete process  $(S_n^\ell)_{n \geq 0}$ , i.e.,  $q_n^\ell(i) = (S_n^\ell(i) + S_n^\ell(i+1))/2$ , and let  $q_n(i)$  denote the corresponding midpoint in the continuous chain  $(S_n)_{n \geq 0}$ , that is,  $q_n(i) = (S_n(i) + S_n(i+1))/2$ . We now show that for all  $i$  we have  $q_n^\ell(i) \leq q_n(i)$  and that  $q_n(i) \leq q_n^\ell(i+1)$  (where  $\leq$  should be interpreted as “is not on the right of”), which implies that the order of the midpoints on the circle is

$$\cdots \leq q_n^\ell(i-1) \leq q_n(i-1) \leq q_n^\ell(i) \leq q_n(i) \leq q_n^\ell(i+1) \leq q_n(i+1) \leq \cdots.$$

First note that, for all  $i$ ,

$$S_n^\ell(i) \leq S_n(i) < S_n^\ell(i+1), \tag{2.2}$$

where the first inequality follows from the fact that  $S_n^\ell(i) = \Theta(S_n(i))$  and the second one since if it were not true then we would have  $S_n^\ell(i) = S_n^\ell(i+1)$ , which cannot happen as initially all the servers are on different locations (i.e., we have by assumption that that  $S_0^\ell(i) \neq S_0^\ell(j)$  for  $i \neq j$ ), and therefore we cannot reach a state with two servers on the same location. The two inequalities  $q_n^\ell(i) \leq q_n(i)$  and  $q_n(i) \leq q_n^\ell(i+1)$  follow immediately from (2.2).

Consider now the request  $r_{n+1}$ . If  $q_n(i-1) < r_{n+1} \leq q_n^\ell(i)$  for some  $i$ , then in both the discrete and the continuous process it will be served by the  $i$ th server and so we will have  $S_{n+1}^\ell = \Theta(S_{n+1})$ . Otherwise (i.e.,

if  $q_n^\ell(i) < r_{n+1} \leq q_n(i)$  for some  $i$ , the discrete process will serve the request with server  $i + 1$  while the continuous process will use server  $i$ , leading to  $S_{n+1}^\ell \neq \Theta(S_{n+1})$ , and the coupling fails. Therefore,

$$\{S_n^\ell = \Theta_\ell(S_n)\} \cap \{S_{n+1}^\ell \neq \Theta_\ell(S_{n+1})\} = \{S_n^\ell = \Theta_\ell(S_n)\} \cap \left( \cup_{i=1}^k \{r_{n+1} \text{ is between } q_n^\ell(i) \text{ and } q_n(i)\} \right),$$

Then, note that  $d(S_n(i), S_n^\ell(i)) < k/\ell$  so that,  $d(q_n(i), q_n^\ell(i)) < k/\ell$ , for all  $0 \leq i \leq k-1$ . Using the definition of  $\delta(k/\ell)$  and the independence of the events,

$$\{S_n^\ell = \Theta_\ell(S_n)\} \quad \text{and} \quad \cup_{i=1}^k \{r_{n+1} \text{ is between } q_n^\ell(i) \text{ and } q_n(i)\},$$

we obtain that,

$$\Pr(r_{n+1} \text{ is between } q_n^\ell(i) \text{ and } q_n(i)) \leq \delta(k/\ell),$$

and, finally,

$$\Pr(S_{n+1}^\ell \neq \Theta_\ell(S_{n+1}) | S_n^\ell = \Theta_\ell(S_n)) \leq k\delta(k/\ell),$$

as required. □

### 2.3. The Case of Two Servers and Uniform Distribution

In the case of uniformly distributed requests for  $k = 2$  servers, we show that the greedy policy is optimal and that the steady-state (expected) cost is  $5/18 = 0.2777\dots$

Consider the greedy algorithm  $G$  and an arbitrary other algorithm  $A$ . Let  $C_n^G$  ( $C_n^A$ ) be the cost incurred by algorithm  $G$  (resp., algorithm  $A$ ) when serving the  $n$ th request, and let  $Z_n^G$  (resp.,  $Z_n^A$ ) be the minimum distance between the servers immediately after serving the  $n$ th request (with  $Z_0 = Z_0^A = Z_0^G = d(S_0(1), S_0(2))$ ).

**Theorem 3.** *If the requests are independent and uniformly distributed on  $\mathbb{T}$ , then the greedy algorithm is optimal in that, for any algorithm  $A$ ,*

$$\mathbb{E} \left[ \sum_n C_n^G \mid Z_0 \right] \leq \mathbb{E} \left[ \sum_n C_n^A \mid Z_0 \right],$$

regardless of the initial position  $Z_0$  of the two servers.

The optimality of the greedy policy for  $k = 2$  follows from the fact that serving each new request with the closest server minimizes the cost to serve the request, while at the same time maximizing the distance between the servers, resulting in a better covering of the space. This is shown in the following lemma.

**Lemma 1.** *For any position of the two servers on the circle and for any request, serving the request with the closest server maximizes the new minimum distance between the two servers.*

*Proof.* Let  $x$  denote the initial distance between the servers and assume, without loss of generality, that the next request occurs on the upper part of the circle, closest to server 1; see Figure 1. We distinguish three cases, depending on the actual position of the request:

1. In  $\Gamma$ : If the request is served by server 1 the new minimum distance will become at least  $x/2$  while if it is served by server 2 it will become at most  $x/2$ .
2. In  $\Delta$ : Assume that the request is at distance  $d$  from server 1. Then, serving the request by server 1 will make the new minimum distance  $d + x$ , while serving it by server 2 will make it  $d$ .
3. In  $E$ : If the request is served by server 1 the new minimum distance will become at least  $1 - x/2$  while if it is served by server 2 it will become at most  $1 - x/2$ .

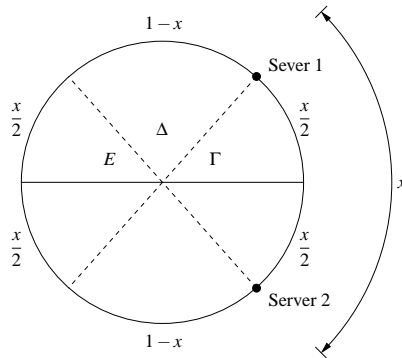


FIGURE 1: An instance of the two servers' position;  $x$  is the current (minimum) distance between the servers.

Therefore, in all three cases the new minimum distance is maximized by serving the new request with the server closest to it.  $\square$

*Proof of Theorem 3.* Assume that at some point the distance between the servers is  $x$  and the next request that arrives is closer to server 1; see Figure 1. Let  $C^C$  be the cost of serving the request with the closest server (server 1), and  $Z^C$  the new server distance after serving the request with server 1.

For  $C^C$  we have:

- With probability  $\frac{x}{2}$ , the new request will occur in  $\Gamma$ , in which case  $C^C \sim \text{Uniform}(0, \frac{x}{2})$ .
- With probability  $1 - \frac{x}{2}$  the new request will occur in  $\Delta \cup E$ , in which case  $C^C \sim \text{Uniform}(0, 1 - \frac{x}{2})$ .

Therefore,

$$\mathbb{E}[C^C | x] = \frac{x}{2} \cdot \frac{x}{4} + \left(1 - \frac{x}{2}\right) \cdot \left(\frac{1}{2} - \frac{x}{4}\right) = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{2}. \quad (2.3)$$

Similarly we can compute:

$$\begin{aligned} \mathbb{E}[Z^C | x] &= \frac{x}{2} \int_0^{\frac{x}{2}} \left(\frac{x}{2} + z\right) \frac{2}{x} dz + (1-x) \int_0^{1-x} (x+z) \frac{1}{1-x} dz + \frac{x}{2} \int_0^{\frac{x}{2}} (1-z) \frac{2}{x} dz \\ &= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2}; \end{aligned} \quad (2.4)$$

$$\begin{aligned} \mathbb{E}[(Z^C)^2 | x] &= \frac{x}{2} \int_0^{\frac{x}{2}} \left(\frac{x}{2} + z\right)^2 \frac{2}{x} dz + (1-x) \int_0^{1-x} (x+z)^2 \frac{1}{1-x} dz + \frac{x}{2} \int_0^{\frac{x}{2}} (1-z)^2 \frac{2}{x} dz \\ &= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{3}. \end{aligned} \quad (2.5)$$



Consider now the  $n$ th request. Using (2.3), the expected cost of the greedy algorithm is computed as,

$$\mathbb{E} [C_n^G | Z_0] = \mathbb{E} [\mathbb{E} [C_n^G | Z_{n-1}^G] | Z_0] = \mathbb{E} \left[ \frac{1}{4}(Z_{n-1}^G)^2 - \frac{1}{2}Z_{n-1}^G + \frac{1}{2} \middle| Z_0 \right], \quad (2.6)$$

and similarly for algorithm  $A$ ,

$$\mathbb{E} [C_n^A | Z_0] = \mathbb{E} [\mathbb{E} [C_n^A | Z_{n-1}^A] | Z_0] \geq \mathbb{E} \left[ \frac{1}{4}(Z_{n-1}^A)^2 - \frac{1}{2}Z_{n-1}^A + \frac{1}{2} \middle| Z_0 \right], \quad (2.7)$$

where the inequality follows from (2.3) and the fact that, conditioned on  $Z_{n-1}^A$ , the cost of serving the  $n$ th request is minimized when it is served by the closest server.

From equations (2.6), (2.7) and Lemma 2 below, it follows that, for every  $n \geq 0$ ,

$$\mathbb{E} [C_n^G | Z_0] \leq \mathbb{E} [C_n^A | Z_0]$$

and, therefore,

$$\mathbb{E} \left[ \sum_n C_n^G \middle| Z_0 \right] \leq \mathbb{E} \left[ \sum_n C_n^A \middle| Z_0 \right],$$

as claimed.  $\square$

**Lemma 2.** For any algorithm  $A$  and for any  $n \geq 0$ ,

$$\mathbb{E} [(Z_n^G)^2 - 2Z_n^G | Z_0] \leq \mathbb{E} [(Z_n^A)^2 - 2Z_n^A | Z_0].$$

*Proof.* The result holds trivially for  $n = 0$ . Assume, inductively, that it holds for some fixed  $n$ . For the  $(n+1)$ st request, applying (2.4) and (2.5), yields,

$$\begin{aligned} \mathbb{E} [(Z_{n+1}^G)^2 - 2Z_{n+1}^G | Z_n^G] &= -\frac{1}{4}(Z_n^G)^2 + \frac{1}{2}Z_n^G + \frac{1}{3} - 2 \left( -\frac{1}{4}(Z_n^G)^2 + \frac{1}{2}Z_n^G + \frac{1}{2} \right) \\ &= \frac{1}{4}(Z_n^G)^2 - \frac{1}{2}Z_n^G - \frac{2}{3} \end{aligned}$$

and hence,

$$\begin{aligned} \mathbb{E} [(Z_{i+n}^G)^2 - 2Z_{i+n}^G | Z_0] &= \mathbb{E} \left[ \mathbb{E} [(Z_{i+n}^G)^2 - 2Z_{i+n}^G | Z_n^G] \middle| Z_0 \right] \\ &= \mathbb{E} \left[ \frac{1}{4}(Z_n^G)^2 - \frac{1}{2}Z_n^G - \frac{2}{3} \middle| Z_0 \right]. \end{aligned} \quad (2.8)$$

Suppose the  $(n+1)$ st request is at some point  $P$ . In order to serve it, the algorithm can either send the closest or the furthest server. Let the lengths of the gaps that will be produced by using the closest or the furthest server be denoted by  $Y_P^C$  and  $Y_P^F$ , respectively. Then, Lemma 1 together with the fact that  $x^2 - 2x$  is decreasing for  $x \in [0, 1]$ , imply that for any  $P$ ,

$$(Y_P^C)^2 - 2Y_P^C \leq (Y_P^F)^2 - 2Y_P^F.$$

Therefore, for any  $Z_n^A$ , the value of,

$$\mathbb{E} [(Z_{n+1}^A)^2 - 2Z_{n+1}^A | Z_n^A],$$

is minimized when the algorithm moves the closest server.

Using again (2.4) and (2.5), we obtain,

$$\mathbb{E} [(Z_{n+1}^A)^2 - 2Z_{n+1}^A \mid Z_n^A] \geq \frac{1}{4}(Z_n^A)^2 - \frac{1}{2}Z_n^A - \frac{2}{3},$$

and,

$$\begin{aligned} \mathbb{E} [(Z_{n+1}^A)^2 - 2Z_{n+1}^A \mid Z_0] &= \mathbb{E} \left[ \mathbb{E} [(Z_{n+1}^A)^2 - 2Z_{n+1}^A \mid Z_n^A] \mid Z_0 \right] \\ &\geq \mathbb{E} \left[ \frac{1}{4}(Z_n^A)^2 - \frac{1}{2}Z_n^A - \frac{2}{3} \mid Z_0 \right]. \end{aligned} \quad (2.9)$$

Combining (2.8), (2.9) and the inductive hypothesis, completes the proof.  $\square$

Finally, we examine the cost distribution induced by the greedy algorithm. Let  $(Y_n)_{n \geq 0}$  denote the Markov chain defined by the clockwise distance between the two servers. Due to the rotation-invariance of the uniform distribution, the following simple recurrence formula holds for the process  $(Y_n)_{n \geq 0}$ ,

$$Y_{n+1} = \frac{1}{2}Y_n + \varepsilon_{n+1}, \quad \text{all } n \geq 1,$$

where the  $(\varepsilon_n)_{n \geq 1}$  are i.i.d. Uniform $[0, 1]$ . This is exactly the statistical setting of an auto-regressive AR(1) process; see, e.g., [8]. In particular, under appropriate assumptions on the innovation sequence  $(\varepsilon_n)_{n \geq 1}$ , the Markov chain  $(Y_n)_{n \geq 0}$  has a unique invariant distribution and it can be shown to satisfy an array of classical limit theorems, including the strong law of large numbers, the central limit theorem, the law of iterated logarithm, and so on. Here we establish some simple properties of the limiting distance  $\lim_n Y_n$ , and we compute the steady-state expected cost.

By induction, for every  $n \geq 1$ ,

$$Y_n = \frac{Y_0}{2^n} + \sum_{k=0}^{n-1} \frac{\varepsilon_{n-k}}{2^k},$$

and, therefore,

$$Z = \lim_{n \rightarrow \infty} Y_n = \sum_{k=0}^{\infty} \frac{\varepsilon_{n-k}}{2^k}.$$

Hence, the characteristic function of  $Z$  satisfies the functional equation,

$$\phi(\theta) = \phi\left(\frac{\theta}{2}\right) \phi_U(\theta) = \prod_{k=0}^{\infty} \phi_U\left(\frac{\theta}{2^k}\right),$$

where  $\phi_U(\theta) = \frac{e^{i\theta} - 1}{i\theta}$  is the characteristic function of the uniform distribution in  $[0, 1]$ .

The expected cost in this case can be computed explicitly as,

$$\mathbb{E}[C] = \mathbb{E}[\mathbb{E}[C|Z]] = \mathbb{E} \left[ \int_0^{Z/2} x \, dx + \int_0^{(2-Z)/2} x \, dx \right] = \frac{1}{4}\mathbb{E}[Z^2] - \frac{1}{2}\mathbb{E}[Z] + \frac{1}{2},$$

and substituting the values of the first and second moments of  $Z$  gives the expected cost as,

$$E(C) = \frac{5}{18} = 0.27777 \dots$$

From the characteristic function of  $Z$  we can also obtain additional information. Noting that, for all  $k \geq 1$ ,  $\phi(\theta)$  can be rewritten as,

$$\phi(\theta) = e^{i\theta} \prod_{k=0}^{\infty} \frac{\sin(\theta/2^{k+1})}{\theta/2^{k+1}},$$

it is easily verified that  $|\theta|^k \phi(\theta)$  is integrable and that then  $Z$  admits a density  $f$  which is infinitely differentiable and satisfies,

$$f(x) = \frac{1}{\pi} \int_0^{+\infty} \cos(\theta(1-x)) \prod_{k=0}^{\infty} \frac{\sin(\theta/2^{k+1})}{\theta/2^{k+1}} d\theta$$

Moreover, the derivative of  $f$  equals,

$$f'(x) = \frac{1}{\pi} \int_0^{+\infty} \theta \sin(\theta(1-x)) \prod_{k=1}^{\infty} \frac{\sin(\theta/2^{k+1})}{\theta/2^{k+1}} d\theta.$$

From these expressions we deduce that, for every  $x \in [0, 2]$ ,  $f(x) = f(2-x)$  and that the function  $f'$  vanishes at  $x = 1$ . A closed formula for the density distribution appears difficult to obtain.

### 3. Open Problems

We are interested in the mean cost to serve a request when it arrives with distribution  $\mu$ . When the  $k$  servers are in position  $\mathbf{p} = (p_1, \dots, p_k)$ , the mean cost to serve a new request is given by,

$$\int_{\mathbb{T}} \left( \min_{i=1, \dots, k} d_0(x, p_i) \right) d\mu(x),$$

and the asymptotic cost is,

$$C(\mu, k) = \int_{\mathbb{T}^k} \int_{\mathbb{T}} \left( \min_{i=1, \dots, k} d_0(x, p_i) \right) d\mu(x) d\pi^k(p_1, \dots, p_k).$$

- Scaling the problem with respect to  $k$ , so that  $k$  servers move on a circle of circumference  $k$ , how does  $C(\mu, k)$  behave as a function on  $k$ ? Figure 2 gives simulation results for the uniform distribution, showing that  $C(\text{uniform}, k)$  is monotonically increasing with  $k$ .
- When  $k = 2$ , is it possible to characterize the steady-state distribution  $\pi$  when  $\mu$  is not the uniform distribution?
- Which parts of the above analysis extend to the case of servers moving on the surface of a ball?

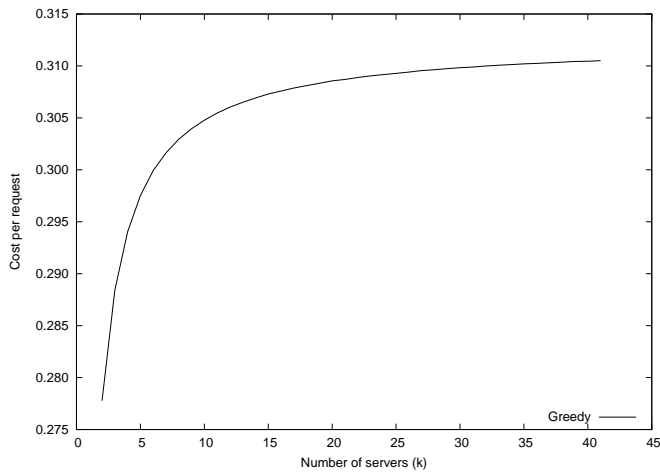


FIGURE 2: The expected cost of serving a request as a function of  $k$ .

### Appendix A. Proof of Theorem 1

Recall that a Markov chain with state space  $\Sigma$  and transition kernel  $P(\cdot, \cdot)$  satisfies *Doebelin's condition* [8, page 391] if there exists a probability measure  $\lambda$  on  $\Sigma$  with the property that for some  $m \geq 1$  and  $\varepsilon \in (0, 1)$ ,

$$P^m(\mathbf{p}, S) \geq \varepsilon \lambda(S),$$

for every  $\mathbf{p} \in \Sigma$  and every (measurable)  $S \subset \Sigma$ .

We define now the probability measure  $\lambda$  that we will use. For  $\delta < 1/8$  arbitrary but fixed, define the intervals  $I_0 = [k - \delta, k) \cup [0, \delta]$  and  $I_i = [i - \delta, i + \delta]$  for  $1 \leq i \leq k - 1$ . Let  $(Y_i)_{0 \leq i \leq k-1}$  be independent random variables, with each  $Y_i$  distributed according to  $\mu(\cdot|I_i)$ . Then  $\lambda$  is the probability measure on  $\Sigma$  induced by the joint distribution of  $Y_0, Y_1, Y_2, \dots, Y_{k-1}$ , so that, in particular, for any set  $S$  of the form  $S = S_0 \times S_1 \times \dots \times S_{k-1} \subset \Sigma$ , where each  $S_i \subset \mathbb{T}$ :

$$\lambda(S) = \prod_{i=0}^{k-1} \frac{\mu(S_i \cap I_i)}{\mu(I_i)}. \quad (\text{A.1})$$

Notice that for the verification of the Doebelin condition it suffices to consider sets of the form  $S = S_0 \times S_1 \times \dots \times S_{k-1} \subset \Sigma$ , where each  $S_i \subset I_i$ .

To that end we need to show that, for any initial position  $\mathbf{p}$  of the servers, there is a sequence of requests that has probability at least  $\varepsilon \lambda(S)$ , such that the greedy algorithm sends every server to a final position  $\mathbf{p}' \in S$  in exactly  $m$  moves. We will establish this by describing a procedure (defined by a sequence of requests) that leads the greedy algorithm to move the servers to  $S$ .

The basic steps of the procedure are the following:

1. Send each server  $i$  to the corresponding interval  $I_i$ ;

2. Perform some additional moves until  $m - k$  moves have been made;
3. Send each server to its final position in  $k$  moves.

Step 2 is necessary, since Doeblin's condition requires that the greedy algorithm reaches the final position in *exactly*  $m$  moves. Below, each step is described and analyzed in detail; first recall that  $\mathbf{p} = (p_0, p_1, \dots, p_{k-1})$ , where each  $p_i$  is the position of the  $i$ th server, and define the clockwise "distance" function,

$$\bar{d}(x, y) = \begin{cases} y - x & \text{if } x \leq y \\ k + y - x & \text{otherwise.} \end{cases}$$

We also define:

- $\bar{d}_i = \bar{d}(p_i, p_{i+1})$ , the distance between servers  $i$  and  $i + 1$ .
- A function  $A$  that maps two points of the circle to the interval between them,

$$A(x, y) = \begin{cases} [x, y) & \text{if } x \leq y \\ [x, k) \cup [0, y) & \text{otherwise.} \end{cases}$$

- The point-to-interval distances,

$$\begin{aligned} \bar{d}(x, I) &= \inf_{y \in I} \bar{d}(x, y) \\ \bar{d}(I, x) &= \inf_{y \in I} \bar{d}(y, x). \end{aligned}$$

Finally, let  $c > 0$  be a constant such that the density  $f$  of  $\mu$  with respect to the Lebesgue measure on  $\mathbb{T}$  satisfies,  $\inf_{x \in \mathbb{T}} f(x) \geq c$ .

### Step 1.

**Lemma 3.** *For any initial position of the servers, the greedy algorithm will send each server  $i$  to  $I_i$  in at most  $12k^2 - 8k$  moves with probability at least  $\left(\frac{c}{4k}\right)^{12k^2 - 9k} \left(\frac{2\delta c}{k}\right)^k$ .*

*Proof.* We describe a procedure defined by a sequence of requests for which the greedy algorithm sends each server  $i$  to  $I_i$ . The probability that this procedure will be followed is at least  $\left(\frac{c}{4k}\right)^{12k^2 - 9k} \left(\frac{2\delta c}{k}\right)^k$ .

Procedure  $\mathcal{P}$ :

- 1.1. Send server 0 to  $I_0$
- 1.2. Send each server  $i$  to  $I_i$

We describe and analyze each step separately.

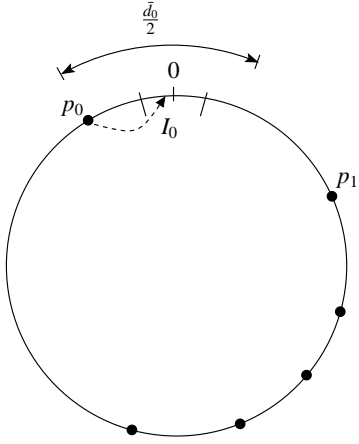
#### Step 1.1

loop

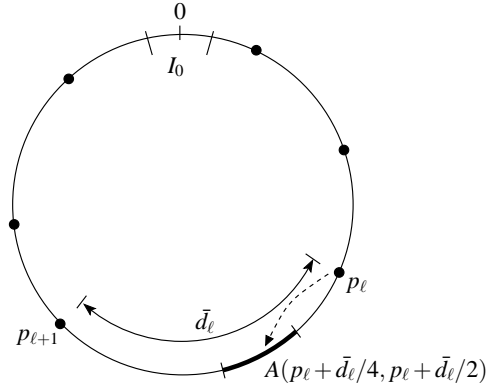
```

if  $p_0 \in I_0$  then
  end
else if  $I_0 \subset A(p_0, p_0 + \bar{d}_0/2)$  then {move of type 1 (see Figure 3(a))}
  send server 0 to  $I_0$  {probability  $\geq (2\delta c)/k$ }
else {move of type 2 (see Figure 3(b))}
   $\ell \leftarrow \arg \max_i \bar{d}_i$  {therefore  $\bar{d}_\ell \geq 1$ }
  send server  $\ell$  to  $A(p_\ell + \bar{d}_\ell/4, p_\ell + \bar{d}_\ell/2)$  {probability  $\geq c/4k$ }
end if
end loop

```



(a) Move of type 1. The interval  $I_0$  is entirely contained in the section of the circle served by server 0 according to the greedy algorithm, so server 0 will move inside  $I_0$  with probability at least  $(2\delta c)/k$ .



(b) Move of type 2. The gap  $\bar{d}_\ell$  is the maximum gap between two consecutive servers, so  $\bar{d}_\ell \geq 1$ . Thus, there is probability at least  $(\bar{d}_\ell c)/4k \geq c/4k$  that a request will fall on the highlighted area  $A(p_\ell + \bar{d}_\ell/4, p_\ell + \bar{d}_\ell/2)$ , decreasing the potential  $\phi$  by at least  $\bar{d}_\ell/4 \geq 1/4$ .

FIGURE 3: Examples of moves in Step 1.1

**Lemma 4.** *In at most  $8k^2 - 4k$  moves and with probability at least  $\left(\frac{c}{4k}\right)^{8k^2 - 4k - 1} \cdot \frac{(2\delta c)}{k}$ , step 1.1 will send server 0 to  $I_0$ .*

*Proof.* We define the following potential function:

$$\phi(\mathbf{p}) = \mathbf{1}_{\{p_0 \notin I_0\}} \cdot \left( k \cdot \bar{d}(p_0, 0) + \sum_{i=1}^{k-1} \bar{d}(p_i, p_0) \right),$$

with  $\mathbf{1}_A$  being the indicator function of event  $A$ . We will show the following facts:

**Fact 1.**  $0 \leq \phi(\mathbf{p}) \leq 2k^2 - k$ .

**Fact 2.**  $\phi(\mathbf{p}) = 0$  if and only if  $p_0 \in I_0$ .

**Fact 3.** At each move,  $\phi(\mathbf{p})$  either becomes 0 with probability at least  $(2\delta c)/k$  and step 1.1 ends, or decreases by at least  $1/4$  with probability at least  $c/4k$ .

Fact 1 holds since  $\mathbf{1}_{\{p_0 \notin I_0\}} \in \{0, 1\}$  and  $0 \leq \bar{d}(\cdot, \cdot) \leq k$ .

Fact 2 is obvious.

In order to show fact 3, we consider the two types of moves of the procedure described earlier. In the case of a type-1 move, the probability to be executed is  $(2\delta c)/k$  and step 1.1 terminates. For a type-2 move we consider the following two cases:

- $\ell \neq 0$ : In this case, notice that the total length of the circle is  $k$  and the total number of gaps is  $k$ . Therefore the length of the largest gap  $\bar{d}_\ell$  is at least 1. Hence we have that the probability of the move is at least  $c/4k$ , and the distance  $\bar{d}(p_\ell, p_0)$  decreases by at least  $1/4$  (with the rest of the distances remaining the same), resulting to a decrease of  $\phi(\mathbf{p})$  by at least  $1/4$ .
- $\ell = 0$ : Again using the previous reasoning, the move is performed with probability at least  $c/4k$ . If server 0 moves into  $I_0$ ,  $\phi(\mathbf{p})$  becomes zero (and notice that the probability  $c/4k \geq (2\delta c)/k$  for  $\delta \leq 1/8$ ) and step 1.1 ends. Otherwise, assume that server 0 moves by  $t \geq 1/4$ . Then  $\bar{d}(p_0, 0)$  decreases by  $t$  and every other distance  $\bar{d}(p_i, p_0)$  increases by  $t$ . So the value of  $\phi(\mathbf{p})$  decreases by:

$$k \cdot t - \sum_{i=1}^{k-1} t = kt - (k-1)t = t \geq 1/4.$$

From the above facts we deduce that the total number of moves that will be performed cannot be more than

$$\frac{2k^2 - k}{\frac{1}{4}} = 8k^2 - 4k,$$

and they all have probability at least  $c/4k$  except possibly for the last one, which has probability at least  $(2\delta c)/k$ . Hence the probability of the first server moving inside  $I_0$  is at least:

$$\left(\frac{c}{4k}\right)^{8k^2 - 4k - 1} \cdot \frac{2\delta c}{k}.$$

### Step 1.2.

To send each server  $i$  to  $I_i$ , we define the following procedure:

Procedure  $\mathcal{P}_2(p_s, p_{s+1}, \dots, p_t)$ :

**Input:**  $p_s, p_{s+1}, \dots, p_t$      $p_s \in I_s, p_t \in I_t$

**Output:**  $p'_s, p'_{s+1}, \dots, p'_t$      $p_i \in I_i, i = s, s+1, \dots, t$

**if**  $t = s$  **then**

$p'_s \leftarrow p_s$

```

end
else if  $t = s + 1$  then
   $p'_s \leftarrow p_s$ 
   $p'_t \leftarrow p_t$ 
  end
else if there exists a  $j, s < j < t$  such that  $p_j \in I_j$  then
   $\mathcal{P}_2(p_s, p_{s+1}, \dots, p_j)$ 
   $\mathcal{P}_2(p_j, p_{j+1}, \dots, p_t)$ 
  end
else
  find a good server  $j$  {to be defined next}
  send server  $j$  to  $I_j$  {using the procedure we describe next}
   $\mathcal{P}_2(p_s, p_{s+1}, \dots, p_j)$ 
   $\mathcal{P}_2(p_j, p_{j+1}, \dots, p_t)$ 
  end
end if

```

We can now define what we mean by a “good” server; we say server  $j$  is good if there are no other servers between server  $j$  and the corresponding interval  $I_j$ , and the rest of the servers are all sufficiently far from  $I_j$ . An example is shown in Figure 4. Formally, we define:

A server  $j$  is *good* if, for every  $i \neq j$ , either  $p_i \notin A(p_j, j)$  and  $\bar{d}(I_j, p_i) > 1 - 2\delta$  (left type), or  $p_i \notin A(j, p_j)$  and  $\bar{d}(p_i, I_j) > 1 - 2\delta$  (right type).

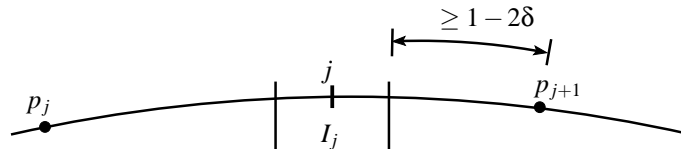


FIGURE 4: An example of a *good* server of left type. There are no servers between server  $j$  (at position  $p_j$ ) and interval  $I_j$ , and the distance (to the right) of all the other servers from  $I_j$  is higher than  $1 - 2\delta$ .

**Lemma 5.** *Let  $0 \leq s < t \leq k \pmod{k}$ , with  $p_s \in A(k - \delta, s + \delta)$  and  $p_t \in A(t - \delta, \delta)$ . Then there exists a good server  $j \in \{s + 1, \dots, t - 1\}$ .*

*Proof.* The proof is by induction on the distance  $t - s$ . The base cases  $t - s = 2$  and  $3$  are easy to verify by enumerating all possible position situations of the server(s) between  $s$  and  $t$ .

Assume now that the lemma holds for all  $r < t - s$ . We will show that it also holds for  $t - s$ . We consider the following three cases:



1.  $p_{s+1} \notin A(k - \delta, s + 1 + \delta)$ : Then notice that there is no server between  $s + 1$  and  $p_{s+1}$ , and we have  $s + 1 - \delta - p_s > s + 1 - \delta - (s + \delta) = 1 - 2\delta$ , and thus,  $\bar{d}(p_s, I_{s+1}) > 1 - 2\delta$  (the same holds trivially for the rest of the servers), which means that server  $s + 1$  is a good server (of right type), and so we select  $j = s + 1$ .
2.  $p_{t-1} \notin A(t - 1 - \delta, \delta)$ : Similarly to the first case, we have that server  $t - 1$  is a good server (of left type) and we select  $j = t - 1$ .
3.  $p_{s+1} \in A(k - \delta, s + 1 + \delta)$  and  $p_{t-1} \in A(t - 1 - \delta, \delta)$ : Then by the induction hypothesis we know that there exists a  $j' \in \{s + 2, \dots, t - 2\}$  that satisfies the conditions and hence we can select  $j = j'$ .

Having proven Lemma 5, we can now describe the procedure that sends the good server  $j$  to  $I_j$ :

```

if  $j$  is a left-type good server then
  loop
    if  $p_j \in I_j$  then
      end
    else if  $I_j \subset A(p_j, p_j + \bar{d}_j/2)$  then
      send server  $j$  to  $I_j$  {probability  $\geq (2\delta c)/k$ }
    else
      send server  $j$  to  $A(p_j + \bar{d}_j/4, p_j + \bar{d}_j/2)$  {probability  $\geq c/4k$ }
    end if
  end loop
else { $j$  is a right-type good server}
  loop
    if  $p_j \in I_j$  then
      end
    else if  $I_j \subset A(p_j - \bar{d}_{j-1}/2, p_j)$  then
      send server  $j$  to  $I_j$  {probability  $(2\delta c)/k$ }
    else
      send server  $j$  to  $A(p_j - \bar{d}_{j-1}/2, p_j - \bar{d}_{j-1}/4)$  {probability  $\geq c/4k$ }
    end if
  end loop
end if

```

Notice that, at each move, either  $j$  enters  $I_j$  (with probability at least  $(2\delta c)/k$ ) or moves by a distance of at least  $\bar{d}_j/4 \geq 1/4$  (or  $\bar{d}_{j-1}/4 \geq 1/4$ ) with probability at least  $c/4k$  and remains a good server. Since the distance between  $j$  and  $I_j$  cannot be more than  $k$ , the total number of moves required to move  $j$  into  $I_j$  is

bounded by  $4k$  and their probability is at least:

$$\left(\frac{c}{4k}\right)^{4k-1} \cdot \frac{2\delta c}{k}.$$

Also note that we can execute the procedure  $\mathcal{P}_2$  for  $s = 0, t = k \pmod{k}$  in order to send all the servers to their corresponding intervals.

**Lemma 6.** *In at most  $4k^2 - 4k$  moves and with probability at least  $\left(\frac{c}{4k}\right)^{(4k-1)(k-1)} \left(\frac{2\delta c}{k}\right)^{k-1}$ , step 1.2 will send each server  $i \neq 0$  to  $I_i$ .*

*Proof.* The procedure sending the good server  $j$  to  $I_j$  will be executed at most  $k - 1$  times. Each time requires at most  $4k$  moves and takes place with probability at most  $\left(\frac{c}{4k}\right)^{4k-1} \cdot \frac{2\delta c}{k}$ .

Combining Lemmas 4 and 6 completes the proof of Lemma 3.

### Step 2.

For any possible initial configuration of the servers and any sequence of requests complying with step 1, the  $k$  moves of step 2 will be performed. However, as mentioned above, in order to satisfy Doebelin's condition we must show that we can reach the final configuration in exactly  $m$  moves, but for step 1 we only gave an upper bound of  $12k^2 - 8k$  for the total number of moves. Hence, additional moves may be required in order to reach exactly that bound. The additional moves that we allow, are induced by requests that are close to the intervals  $I_i$ . In particular, they may fall in

$$\left(k - \frac{1}{4}, k\right) \cup \left[0, \frac{1}{4}\right) \cup \left[\bigcup_{i=1}^{k-1} \left(i - \frac{1}{4}, i + \frac{1}{4}\right)\right]$$

It is easy to see that for any sequence of requests that falls in the above set, none of the servers moves far away from its interval, which will allow them at the end to move close to their final positions.

Each request takes place with probability  $c/2$ , which is higher than any single request of step 1, and consequently the lower bound on the probability that we gave at step 1, holds even when fewer than  $12k^2 - 8k$  moves are performed during step 1 and the rest are performed during the current step.

### Step 3.

The last  $k$  requests will send the servers to a final configuration in  $S = S_0 \times S_1 \times \cdots \times S_{k-1}$ . Specifically, every server  $i$  will enter  $S_i \subset I_i$ , as we argued in step 2.

Consider now the probability that the last  $k$  requests send each server  $i$  to  $S_i$ . The total number of permutations for the order to send the servers is  $k!$  and for each of them the probability that it happens is,  $\prod_{i=0}^{k-1} \mu(S_i)$ , so the total probability is

$$k! \prod_{i=0}^{k-1} \mu(S_i) \geq k! \prod_{i=0}^{k-1} \mu(S_i \cap I_i) \geq k! \left(\frac{2\delta c}{k}\right)^k \cdot \prod_{i=0}^{k-1} \frac{\mu(S_i \cap I_i)}{\mu(I_i)} = k! \left(\frac{2\delta c}{k}\right)^k \lambda(S),$$

using (A.1).

Taking into account all the moves of the three steps and the corresponding probabilities, we have that Doeblin's condition holds for

$$m = 12k^2 - 7k$$

and

$$\varepsilon = k! \left(\frac{c}{4k}\right)^{12k^2-9k} \left(\frac{2\delta c}{k}\right)^{2k}.$$

### Acknowledgements

The authors are grateful to Guillaume Aubrun and Francis Comets for helpful discussions. IK was supported in part by a Marie Curie International Outgoing Fellowship, PIOF-GA-2009-235837. CD and NG-P was supported by CNRS grant DREI 21514.

### References

- [1] Billingsley, P. *Convergence of Probability Measures*. Wiley, New York, 1968.
- [2] A. Borodin and R. El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, Cambridge, 1998.
- [3] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Second edition. Springer-Verlag, New York, 1998.
- [4] I. Kontoyiannis and S. P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Annals of Applied Probability*, 13:304-362, February 2003.
- [5] E. Koutsoupias and C. H. Papadimitriou. On the  $k$ -server conjecture. *Journal of the ACM*, 42(5):971–983, Sept. 1995.
- [6] M. Manasse, L. McGeoch, and D. Sleator. Competitive algorithms for on-line problems. In R. Cole, editor, *Proceedings of the 20th Annual ACM Symposium on the Theory of Computing*, pages 322–333, Chicago, IL, May 1988. ACM Press.
- [7] M. S. Manasse, L. A. McGeoch, and D. D. Sleator. Competitive algorithms for server problems. *Journal of Algorithms*, 11(2):208–230, June 1990.
- [8] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, London, New York, 1993.
- [9] Norris J.R. *Markov Chains*. Cambridge Series in Statistical and Probabilistic Mathematics, 1998.