Algorithmic Game Theory
Efficiency at an Equilibrium

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Based on slides by Alexandros Voudouris

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- Now, we can ask the following questions: which state of the game minimizes the social cost? Is it an equilibrium? If not, then what is the difference between the social cost of an equilibrium and the minimum possible social cost?

Load balancing: Example 1

- Two players and two machines with latencies $f_1(x) = x$ and $f_2(x) = (2 + \epsilon)x$, where ϵ is a very small positive constant (like $\epsilon = 0.0001$)

	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{M_1}$	$\mathbf{2, 2}$	$\mathbf{1, 2 + \epsilon}$
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- The states (M_1, M_2) and (M_2, M_1) however are the optimal ones with social cost $3 + \epsilon$
- The strategic behavior of the players does not allow them to reach the optimal state of the game

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$$\mathrm{PoS} = \min_{\boldsymbol{s} \in \mathrm{NE}} \frac{\mathrm{SC}(\boldsymbol{s})}{\mathrm{SC}(\boldsymbol{s}_{OPT})} \qquad \mathrm{PoA} = \max_{\boldsymbol{s} \in \mathrm{NE}} \frac{\mathrm{SC}(\boldsymbol{s})}{\mathrm{SC}(\boldsymbol{s}_{OPT})}$$

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- The price of stability is an *optimistic* measure: it considers the best equilibrium (with minimum social cost)
- The price of anarchy is a *pessimistic* measure: it considers the worst equilibrium (with maximum social cost)

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- (M_1, M_1) is the only equilibrium of the game, with social cost 4
- The states (M_1, M_2) and (M_2, M_1) are the optimal ones with social cost $3+\epsilon$

$$\text{PoS} = \text{PoA} = \frac{4}{3+\epsilon}$$

Load balancing: Example 2

- Change the latency of the second machine to $f_2(x) = (2 - \epsilon)x$

	M_1	M_2
M_1	$2, 2$	$1, 2-\epsilon$
M_2	$2-\epsilon, 1$	$4-2\epsilon, 4-2\epsilon$

- (M_1, M_2) and (M_2, M_1) are both equilibrium states and have optimal social cost of $3 - \epsilon$

$$\text{PoS} = \text{PoA} = \frac{3-\epsilon}{3-\epsilon} = 1$$

Load balancing: Example 3

- Change the latency of the second machine to $f_2(x) = 2x$

	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{M_1}$	$\mathbf{2, 2}$	$\mathbf{1, 2}$
$\boldsymbol{M_2}$	$\mathbf{2, 1}$	$\mathbf{4, 4}$

- There are three equilibrium states: (M_1, M_1), (M_1, M_2) and (M_2, M_1)
- (M_1, M_1) has social cost 4, while (M_1, M_2) and (M_2, M_1) have social cost 3 and are the optimal states

$$\text{PoS} = \frac{3}{3} = 1 \qquad \text{PoA} = \frac{4}{3}$$

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  - $n_e(\boldsymbol{s})$ is the load of e, equal to the number of players using it

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- We will show bounds on the price of stability and the price of anarchy for this special class of congestion games
- We want these bounds to be close to 1 to guarantee high efficiency

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$$\mathrm{SC}(\boldsymbol{s}) \leq \frac{1}{\lambda} \cdot \Phi(\boldsymbol{s}) \leq \frac{1}{\lambda} \cdot \Phi(\boldsymbol{s}_{OPT}) \leq \frac{\mu}{\lambda} \cdot \mathrm{SC}(\boldsymbol{s}_{OPT}) \Rightarrow \text{PoS} \leq \frac{\mu}{\lambda}$$

Linear congestion games: PoS

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- All we need to show is that there exist parameters λ and μ such that $\mu/\lambda = 2$
- In particular we will show that $\lambda = 1/2$ and $\mu = 1$:

$$\frac{1}{2} \cdot \mathrm{SC}(\boldsymbol{s}) \leq \Phi(\boldsymbol{s}) \leq \mathrm{SC}(\boldsymbol{s})$$

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$$\Phi(\boldsymbol{s}) \geq \sum_{e \in E} \left(a_e \frac{n_e(\boldsymbol{s})^2}{2} + \frac{1}{2} \cdot b_e n_e(\boldsymbol{s}) \right) = \frac{1}{2} \cdot \mathrm{SC}(\boldsymbol{s})$$

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- Since $n_e(\boldsymbol{s}) \leq n_e(\boldsymbol{s})^2$, we get

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A general technique for PoA bounds

- Recall that a state $\boldsymbol{s} = (s_1, \dots, s_n)$ is an equilibrium if for each player i the strategy s_i minimizes her personal cost, given the strategies of the other players

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- We have one such inequality for every player

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- We can get an upper bound of λ on the price of anarchy if there exists a strategy y_i for every player i such that

$$\sum_{i \in N} \mathrm{cost}_i(y_i, \boldsymbol{s}_{-i}) \leq \lambda \cdot \mathrm{SC}(\boldsymbol{s}_{OPT})$$

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- The goal is to pinpoint the strategy y_i for each player i, which will allow us to prove an inequality like this

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$$\begin{aligned}
\mathrm{SC}(\boldsymbol{s}) &= \sum_{i \in N} \mathrm{cost}_i(s_i, \boldsymbol{s}_{-i}) \\
&\leq \sum_{i \in N} \mathrm{cost}_i(y_i, \boldsymbol{s}_{-i}) \\
&= \sum_{e \in E} \sum_{i \in N : e \in y_i} (a_e \cdot n_e(y_i, \boldsymbol{s}_{-i}) + b_e)
\end{aligned}$$

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&= \sum_{e \in E} \left(a_e \cdot n_e(\boldsymbol{y}) \cdot (n_e(\boldsymbol{s}) + 1) + b_e n_e(\boldsymbol{y})\right)
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- Since this holds for any $\boldsymbol{y}$, it also holds for $\boldsymbol{s}_{OPT}$ ▢

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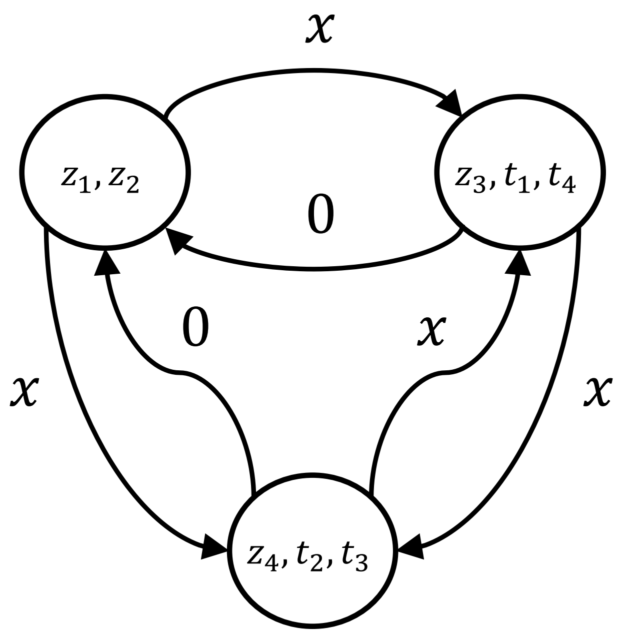
- To show a lower bound, it suffices to construct a specific instance and prove that the social cost of the equilibrium is 5/2 times the optimal social cost

Can we do any better?



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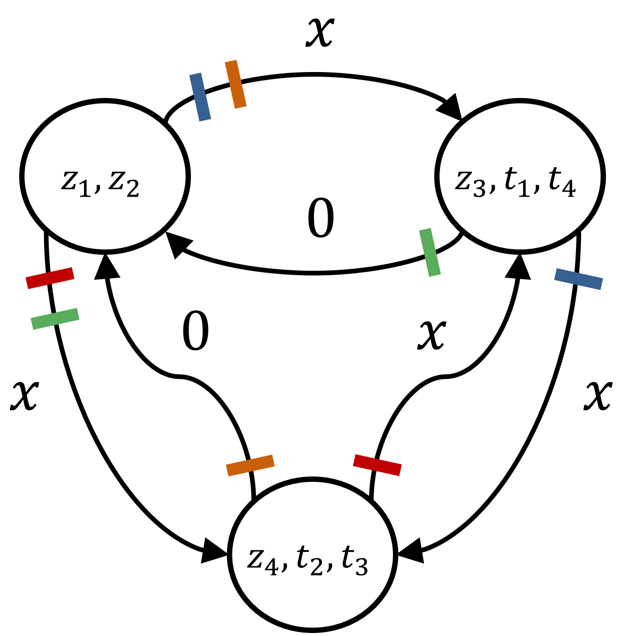
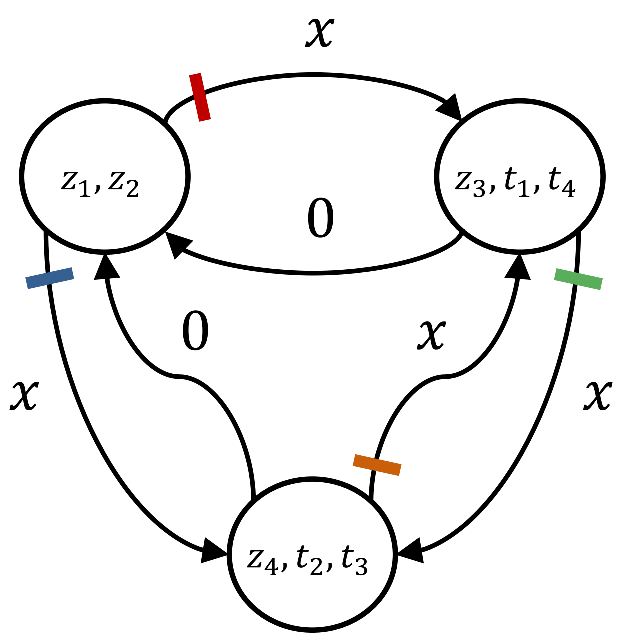


- Equilibrium: each player i uses two edges to connect z_i to t_i
- Players 1 and 2 (red, blue) have cost 3, while players 3 and 4 (green, orange) have cost 2
- By changing to the direct edge, all players would still have the same cost, so there is no reason for them to deviate

Can we do any better?



- Optimal: each player i uses the direct edge between z_i and t_i
- All players have cost 1
- SC(equilibrium) $= 10$ vs. SC(optimal) $= 4$ ⇨ PoA = 5/2

□

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- **PoA of linear congestion games:** tight bound of $5/2$

Some further readings

- **The price of anarchy of finite congestion games**
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  - T. Roughgarden
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  - E. Anshelevich, A. Dasgupta, J. M. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden
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