Computational Social Choice
Distortion

Georgios Birmpas
birbas@diag.uniroma1.it

Based on slides by Alexandros Voudouris

The setting

- A set of agents N and a set of alternatives A

The setting

- A set of agents N and a set of alternatives A

- Each agent $i \in N$ has a **value** v_{ix} for every alternative $x \in A$
- **Unit-sum assumption:** $\sum_{j \in A} v_{ix} = 1$
- **Valuation profile:** $\boldsymbol{v} = (v_{ix})_{i \in N,\, x \in A}$

The setting

- A set of agents N and a set of alternatives A

- Each agent $i \in N$ has a **value** v_{ix} for every alternative $x \in A$
- **Unit-sum assumption:** $\sum_{j\in A} v_{ix} = 1$
- **Valuation profile:** $\boldsymbol{v} = (v_{ix})_{i\in N,\, x\in A}$

- The values of an agent i for the alternatives define a ranking $\succ_i$ over them such that $x \succ_i y$ when $v_{ix} \geq v_{iy}$
  - Ties are broken according to some (fixed) tie-breaking rule
- **Ordinal profile** induced by a valuation profile: $\succ_{\boldsymbol{v}} = (\succ_i)_{i\in N}$

The setting

- An ordinal profile can be induced by many different valuation profiles

The setting

- An ordinal profile can be induced by many different valuation profiles

agent	a	b	c	d
1	0.75	0.15	0.07	0.03
2	0.25	0.15	0.2	0.4
3	0.1	0	0.4	0.5
4	0.21	0.3	0.2	0.29

agent	a	b	c	d
1	0.95	0.03	0.02	0
2	0.25	0.21	0.24	0.3
3	0.24	0.21	0.25	0.3
4	0.02	0.95	0	0.03

The setting

- An ordinal profile can be induced by many different valuation profiles

agent	a	b	c	d
1	0.75	0.15	0.07	0.03
2	0.25	0.15	0.2	0.4
3	0.1	0	0.4	0.5
4	0.21	0.3	0.2	0.29

agent	a	b	c	d
1	0.95	0.03	0.02	0
2	0.25	0.21	0.24	0.3
3	0.24	0.21	0.25	0.3
4	0.02	0.95	0	0.03

agent	ranking			
1	a	b	c	d
2	d	a	c	b
3	d	c	a	b
4	b	d	a	c

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

$$\mathrm{SW}(x|\boldsymbol{v}) = \sum_{i \in N} v_{ix}$$

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

$$\mathrm{SW}(x|\boldsymbol{v}) = \sum_{i \in N} v_{ix}$$

- The social welfare is a measure of how good an alternative is for the society

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

$$\mathrm{SW}(x|\boldsymbol{v}) = \sum_{i \in N} v_{ix}$$

- The social welfare is a measure of how good an alternative is for the society ⇨ **our goal is to choose the alternative with maximum social welfare**

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

$$\mathrm{SW}(x|\boldsymbol{v}) = \sum_{i \in N} v_{ix}$$

- The social welfare is a measure of how good an alternative is for the society ⇨ **our goal is to choose the alternative with maximum social welfare**
- If we had access to the valuation profile, we could obviously make the optimal social choice

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

$$\mathrm{SW}(x|\boldsymbol{v}) = \sum_{i \in N} v_{ix}$$

- The social welfare is a measure of how good an alternative is for the society ⇨ **our goal is to choose the alternative with maximum social welfare**
- If we had access to the valuation profile, we could obviously make the optimal social choice
- But … choices are made by voting rules that have access only to the ordinal profile, and therefore electing the optimal alternative is not an easy task

Social welfare and voting rules

- Question: How efficient is the choice made by a voting rule R?

Social welfare and voting rules

- Question: How efficient is the choice made by a voting rule R?
- The **distortion** of R is the worst-case ratio (over all valuation profiles) between the maximum social welfare (achieved by any alternative) and the social welfare of the alternative chosen by R

$$\mathrm{dist}(R) = \sup_{\boldsymbol{v}} \frac{\max_{x \in A} \mathrm{SW}(x|\boldsymbol{v})}{\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})}$$

Social welfare and voting rules

- Question: How efficient is the choice made by a voting rule R?
- The **distortion** of R is the worst-case ratio (over all valuation profiles) between the maximum social welfare (achieved by any alternative) and the social welfare of the alternative chosen by R

$$\mathrm{dist}(R) = \sup_{\boldsymbol{v}} \frac{\max_{x \in A} \mathrm{SW}(x|\boldsymbol{v})}{\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})}$$

- If $\mathrm{dist}(R) = 1$ then R is optimal, and always chooses the alternative that maximizes the social welfare
- In general however: $\mathrm{dist}(R) \geq 1$

Social welfare and voting rules

- Question: How efficient is the choice made by a voting rule R?
- The **distortion** of R is the worst-case ratio (over all valuation profiles) between the maximum social welfare (achieved by any alternative) and the social welfare of the alternative chosen by R

$$\mathrm{dist}(R) = \sup_{\boldsymbol{v}} \frac{\max_{x \in A} \mathrm{SW}(x|\boldsymbol{v})}{\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})}$$

- If $\mathrm{dist}(R) = 1$ then R is optimal, and always chooses the alternative that maximizes the social welfare
- In general however: $\mathrm{dist}(R) \geq 1$
- We are interested in bounding the distortion of voting rules, and we want these bounds to be as small as possible

A first lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m)$

A first lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m)$

# agents	ranking				
$m/2$	x	y	a_1	...	a_{m-2}
$m/2$	y	x	a_1	...	a_{m-2}

A first lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m)$

# agents	**ranking**				
$m/2$	x	y	a_1	...	a_{m-2}
$m/2$	y	x	a_1	...	a_{m-2}

- R will choose either alternative x or alternative y
- All other alternatives are dominated by these two alternatives

A first lower bound

- Without loss of generality, assume that R chooses alternative x

A first lower bound

- Without loss of generality, assume that R chooses alternative x

# agents	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{a_1}$	...	$\boldsymbol{a_{m-2}}$
$m/2$	$1/m$	$1/m$	$1/m$	...	$1/m$
$m/2$	0	1	0	...	0

A first lower bound

- Without loss of generality, assume that R chooses alternative x

# agents	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{a_1}$	…	$\boldsymbol{a_{m-2}}$
$m/2$	$1/m$	$1/m$	$1/m$	…	$1/m$
$m/2$	0	1	0	…	0

$$\mathrm{SW}(x|\boldsymbol{v}) = \frac{m}{2} \cdot \frac{1}{m} = \frac{1}{2}$$

$$\mathrm{SW}(y|\boldsymbol{v}) = \frac{m}{2} \cdot \left(1 + \frac{1}{m}\right) = \frac{m+1}{2}$$

A first lower bound

- Without loss of generality, assume that R chooses alternative x

# agents	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{a_1}$	...	$\boldsymbol{a_{m-2}}$
$m/2$	$1/m$	$1/m$	$1/m$	...	$1/m$
$m/2$	0	1	0	...	0

$$\left.\begin{array}{l} \mathrm{SW}(x|\boldsymbol{v}) = \dfrac{m}{2} \cdot \dfrac{1}{m} = \dfrac{1}{2} \\[2ex] \mathrm{SW}(y|\boldsymbol{v}) = \dfrac{m}{2} \cdot \left(1 + \dfrac{1}{m}\right) = \dfrac{m+1}{2} \end{array}\right\} \mathrm{dist}(R) \geq \frac{\mathrm{SW}(y|\boldsymbol{v})}{\mathrm{SW}(x|\boldsymbol{v})} = m + 1$$

$\square$

A stronger lower bound

Theorem

The distortion of any deterministic voting rule is $\Omega(m^2)$

A stronger lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m^2)$

- Instance with $n = m(m-2)$ agents
- Alternatives $A = \{x, y, a_1, \ldots, a_{m-2}\}$

A stronger lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m^2)$

- Instance with $n = m(m-2)$ agents
- Alternatives $A = \{x, y, a_1, \ldots, a_{m-2}\}$

- For every $j \in [m-2]$, alternative a_i appears first in m rankings
- Alternative x appears second in $\frac{n}{2} = \Theta(m^2)$ rankings
- Alternative y appears second in $\frac{n}{2} = \Theta(m^2)$ rankings
- All agents that rank first the same alternative a_j, rank second either x or y

A stronger lower bound

- **Case I:** The voting rule chooses alternative a_j for some $j \in [m-2]$

A stronger lower bound

- **Case I:** The voting rule chooses alternative a_j for some $j \in [m-2]$
- Valuation profile $\boldsymbol{v}$:
  - The m agents that rank a_j first have value $1/m$ for all alternatives; assume these agents rank x second
  - All other agents have value $1/2$ for the alternatives they rank at the first two positions

A stronger lower bound

- **Case I:** The voting rule chooses alternative a_j for some $j \in [m-2]$
- Valuation profile $\boldsymbol{v}$:
  - The m agents that rank a_j first have value $1/m$ for all alternatives; assume these agents rank x second
  - All other agents have value $1/2$ for the alternatives they rank at the first two positions

$$\left.\begin{array}{l} \mathrm{SW}(a_j|\boldsymbol{v}) = m \cdot \dfrac{1}{m} = 1 \\[2em] \mathrm{SW}(y|\boldsymbol{v}) = \Theta(m^2) \cdot \dfrac{1}{2} = \Theta(m^2) \end{array}\right\} \quad \mathrm{dist}(R) = \Omega(m^2)$$

A stronger lower bound

- **Case II:** The voting rule chooses x or y

A stronger lower bound

- **Case II:** The voting rule chooses x or y
- Valuation profile $\boldsymbol{v}'$:
  - All agents have value 1 for their favorite alternative a_j, and 0 for everyone else

A stronger lower bound

- **Case II:** The voting rule chooses x or y
- Valuation profile $\boldsymbol{v}'$:
  - All agents have value 1 for their favorite alternative a_j, and 0 for everyone else

$$\left.\begin{array}{l} \mathrm{SW}(x|\boldsymbol{v}') = 0 \\ \mathrm{SW}(y|\boldsymbol{v}') = 0 \\ \mathrm{SW}(z|\boldsymbol{v}') > 0, \forall z \neq x, y \end{array}\right\} \quad \mathrm{dist}(R) \text{ is unbounded}$$

$\square$

An asymptotically tight upper bound

Theorem
There exists a voting rule with distortion $O(m^2)$

An asymptotically tight upper bound

Theorem
There exists a voting rule with distortion $O(m^2)$

- **Plurality rule**

An asymptotically tight upper bound

Theorem
There exists a voting rule with distortion $O(m^2)$

- **Plurality rule**
- The winner x must be ranked first at least n/m times
- The corresponding agents must have value at least $1/m$ for x

An asymptotically tight upper bound

Theorem
There exists a voting rule with distortion $O(m^2)$

- **Plurality rule**
- The winner x must be ranked first at least n/m times
- The corresponding agents must have value at least $1/m$ for x
- Each agent has value at most 1 for the optimal alternative y

An asymptotically tight upper bound

Theorem

There exists a voting rule with distortion $O(m^2)$

- **Plurality rule**
- The winner x must be ranked first at least n/m times
- The corresponding agents must have value at least $1/m$ for x
- Each agent has value at most 1 for the optimal alternative y

$$\left.\begin{aligned} \mathrm{SW}(x|\boldsymbol{v}) &\geq \frac{n}{m}\cdot\frac{1}{m} = \frac{n}{m^2} \\ \mathrm{SW}(y|\boldsymbol{v}) &\leq n \end{aligned}\right\} \quad \mathrm{dist}(\mathbf{PL}) = O(m^2)$$

$\square$

Randomized voting rules

- A randomized voting rule R defines a probability distribution p_R over the alternatives according to which the winning alternative is chosen

Randomized voting rules

- A randomized voting rule R defines a probability distribution p_R over the alternatives according to which the winning alternative is chosen
- The efficiency of R is now measured by the expected social welfare of the winner:

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] = \sum_{x \in A} p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v})$$

Randomized voting rules

- A randomized voting rule R defines a probability distribution p_R over the alternatives according to which the winning alternative is chosen
- The efficiency of R is now measured by the expected social welfare of the winner:

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] = \sum_{x \in A} p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v})$$

- Refinement of distortion:

$$\mathrm{dist}(R) = \sup_{\boldsymbol{v}} \frac{\max_{x \in A} \mathrm{SW}(x|\boldsymbol{v})}{\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})]}$$

An improved distortion bound

Theorem

There exists a randomized voting rule with distortion $O\left(\sqrt{m \cdot \ln m}\right)$

An improved distortion bound

Theorem

There exists a randomized voting rule with distortion $O\left(\sqrt{m \cdot \ln m}\right)$

- Harmonic scoring rule: $\mathbf{H} = (1, 1/2, \dots, 1/m)$
- $sc(x)$ = score of alternative x according to $\mathbf{H}$

An improved distortion bound

Theorem

There exists a randomized voting rule with distortion $O\left(\sqrt{m \cdot \ln m}\right)$

- Harmonic scoring rule: $\mathbf{H} = (1, 1/2, \dots, 1/m)$
- $sc(x)$ = score of alternative x according to $\mathbf{H}$
- **Voting rule:**
  - Rule 1: Choose alternative x with probability $\frac{\mathbf{sc}(x)}{\sum_{y \in A} \mathbf{sc}(y)}$
  - Rule 2: Choose alternative x with probability $1/m$
  - Run the two rules with probability $1/2$ each

An improved distortion bound

- Let x be the optimal alternative
- We distinguish between two cases, depending on the harmonic score of x
  - **Case I:** $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$
  - **Case II:** $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

An improved distortion bound

Case I: $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$

An improved distortion bound

Case I: $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$

- $\sum_{y \in A} sc(y) = n \cdot \sum_{k=1}^{m} \frac{1}{k} \leq n\,(\ln m + 1)$

An improved distortion bound

Case I: $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$

- $\sum_{y \in A} sc(y) = n \cdot \sum_{k=1}^{m} \frac{1}{k} \leq n\,(\ln m + 1)$
- $p_R(x) \geq \frac{1}{2} \cdot \frac{\mathrm{sc}(x)}{\sum_{y \in \mathrm{A}} \mathrm{sc}(y)} \geq \frac{1}{2} \cdot \frac{n \cdot \sqrt{\frac{\ln m+1}{m}}}{n\,(\ln m+1)} = \frac{1}{2\sqrt{m(\ln m+1)}}$

An improved distortion bound

Case I: $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$

- $\sum_{y \in A} sc(y) = n \cdot \sum_{k=1}^{m} \frac{1}{k} \leq n\,(\ln m + 1)$

- $p_R(x) \geq \frac{1}{2} \cdot \frac{\mathbf{SC}(x)}{\sum_{y \in \mathrm{A}} \mathbf{SC}(y)} \geq \frac{1}{2} \cdot \frac{n \cdot \sqrt{\frac{\ln m+1}{m}}}{n\,(\ln m+1)} = \frac{1}{2\sqrt{m(\ln m+1)}}$

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] \geq p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v}) \geq \frac{\mathrm{SW}(x|\boldsymbol{v})}{2\sqrt{m(\ln m + 1)}}$$

An improved distortion bound

Case I: $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m + 1}{m}}$

- $\sum_{y \in A} sc(y) = n \cdot \sum_{k=1}^{m} \frac{1}{k} \leq n\,(\ln m + 1)$
- $p_R(x) \geq \frac{1}{2} \cdot \frac{\mathrm{sc}(x)}{\sum_{y \in \mathrm{A}} \mathrm{sc}(y)} \geq \frac{1}{2} \cdot \frac{n \cdot \sqrt{\frac{\ln m+1}{m}}}{n\,(\ln m + 1)} = \frac{1}{2\sqrt{m(\ln m + 1)}}$

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] \geq p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v}) \geq \frac{\mathrm{SW}(x|\boldsymbol{v})}{2\sqrt{m(\ln m + 1)}}$$

$$\Rightarrow \mathrm{dist}(R) \leq 2\sqrt{m(\ln m + 1)}$$

An improved distortion bound

Case II: $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

An improved distortion bound

Case II: $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

- If alternative x is ranked k–th by agent i, then $v_{ix} \leq \frac{1}{k}$
  ⇨ $\mathrm{SW}(x|\boldsymbol{v}) \leq \mathrm{sc}(x)$

An improved distortion bound

Case II: $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

- If alternative x is ranked k–th by agent i, then $v_{ix} \leq \frac{1}{k}$
  ⇨ $\mathrm{SW}(x|\boldsymbol{v}) \leq \mathrm{sc}(x)$
- For every alternative $y \in A$: $p_R(y) \geq \frac{1}{2m}$

An improved distortion bound

Case II: $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

- If alternative x is ranked k–th by agent i, then $v_{ix} \leq \frac{1}{k}$
  ⇨ $\mathrm{SW}(x|\boldsymbol{v}) \leq \mathrm{sc}(x)$
- For every alternative $y \in A$: $p_R(y) \geq \frac{1}{2m}$

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] \geq \sum_{y \in A} p_R(y) \cdot \mathrm{SW}(y|\boldsymbol{v}) \geq \frac{1}{2m} \cdot \sum_{y \in A} \mathrm{SW}(y|\boldsymbol{v}) = \frac{n}{2m}$$

An improved distortion bound

Case II: $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

- If alternative x is ranked k–th by agent i, then $v_{ix} \leq \frac{1}{k}$
  ⇨ $\mathrm{SW}(x|\boldsymbol{v}) \leq \mathrm{sc}(x)$
- For every alternative $y \in A$: $p_R(y) \geq \frac{1}{2m}$

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] \geq \sum_{y \in A} p_R(y) \cdot \mathrm{SW}(y|\boldsymbol{v}) \geq \frac{1}{2m} \cdot \sum_{y \in A} \mathrm{SW}(y|\boldsymbol{v}) = \frac{n}{2m}$$

$$\Rightarrow \mathrm{dist}(R) = \frac{\mathrm{SW}(x|\boldsymbol{v})}{\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})]} \leq \frac{n \cdot \sqrt{\frac{\ln m + 1}{m}}}{\frac{n}{2m}} = 2\sqrt{m(\ln m + 1)}$$

□

Best known bounds

- We can actually do a little bit better than $\mathrm{O}\left(\sqrt{m \cdot \ln m}\right)$

Theorem
There exists a randomized voting rule with distortion $\mathrm{O}(\sqrt{m}\log^* m)$

Best known bounds

- We can actually do a little bit better than $O\left(\sqrt{m \cdot \ln m}\right)$

Theorem
There exists a randomized voting rule with distortion $O(\sqrt{m}\log^* m)$

- But, not that much better …

Theorem
The distortion of any randomized voting rule is $\Omega(\sqrt{m})$

Summary

Summary

- Agents have implicit values for the alternatives

Summary

- Agents have implicit values for the alternatives
- These values induce the preference rankings
  - Many different valuation profiles can induce the same ordinal profile

Summary

- Agents have implicit values for the alternatives
- These values induce the preference rankings
  - Many different valuation profiles can induce the same ordinal profile
- **Distortion:** worst case ratio over all valuation profiles between the social welfare of the optimal outcome over the social welfare of the outcome chosen by the voting rule

Summary

- Agents have implicit values for the alternatives
- These values induce the preference rankings
  - Many different valuation profiles can induce the same ordinal profile
- **Distortion:** worst case ratio over all valuation profiles between the social welfare of the optimal outcome over the social welfare of the outcome chosen by the voting rule
- **Deterministic rules:** distortion is $\Omega(m^2)$

Summary

- Agents have implicit values for the alternatives
- These values induce the preference rankings
  - Many different valuation profiles can induce the same ordinal profile
- **Distortion:** worst case ratio over all valuation profiles between the social welfare of the optimal outcome over the social welfare of the outcome chosen by the voting rule
- **Deterministic rules:** distortion is $\Omega(m^2)$
- **Randomized rules:** distortion is between $\Omega(\sqrt{m})$ and $O(\sqrt{m}\log^* m)$

Some further readings

- **The distortion of cardinal preferences in voting**
  - A. D. Procaccia and J. S. Rosenschein
  - 10th Workshop on Cooperative Information Agents (CIA), pp. 317-331, 2006
- **Optimal social choice functions: A utilitarian view**
  - C. Boutilier, I. Caragiannis, S. Haber, T. Lu, A. D. Procaccia, and O. Sheffet
  - Artificial Intelligence, vol. 227, pp. 190-213, 2015
- **Subset selection via implicit utilitarian voting**
  - I. Caragiannis, S. Nath, A. D. Procaccia, and N. Shah
  - Journal of Artificial Intelligence Research, vol. 58, pp. 123-152, 2017
- **Approximating optimal social choice under metric preferences**
  - E. Anshelevich, O. Bhardwaj, E. Elkind, J. Postl, P. Skowron
  - Artificial Intelligence, vol. 264, pp. 27-51, 2018