Algorithmic Game Theory

Solution concepts in games, existence of Nash Equilibria, and quality metrics

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Based on slides by Vangelis Markakis and Alexandros Voudouris

Existence of Nash equilibria

Nash equilibria: Recap

Recall the problematic issues we have identified for pure Nash equilibria:

1. Non-existence: there exist games that do not possess an equilibrium with pure strategies
2. Non-uniqueness: there are games that have many Nash equilibria
3. Welfare guarantees: The equilibria of a game do not necessarily have the same utility for the players

Have we made any progress by considering equilibria with mixed strategies?

Existence of Nash equilibria

- Theorem [Nash 1951]: Every finite game possesses at least one equilibrium when we allow mixed strategies
  - Finite game: finite number of players, and finite number of pure strategies per player
- Corollary: if a game does not possess an equilibrium with pure strategies, then it definitely has one with mixed strategies
- One of the most important results in game theory
- Nash's theorem resolves the issue of non-existence
  - By allowing a richer strategy space, existence is guaranteed, no matter how big or complex the game might be

Examples

- In Prisoner's dilemma or Bach-or-Stravinsky, there exist equilibria with pure strategies
  - For such games, Nash's theorem does not add any more information. However, in addition to pure equilibria, we may also have some mixed equilibria
- Matching-Pennies: For this game, Nash's theorem guarantees that there exists an equilibrium with mixed strategies
  - In fact, it is the profile we saw: ((1/2, 1/2), (1/2, 1/2))
- Rock-Paper-Scissors?
  - Again the uniform distribution: ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))

Nash equilibria: Computation

- Nash's theorem only guarantees the existence of Nash equilibria
  - Proof reduces to using Brouwer's fixed point theorem
- Brouwer's theorem: Let $f: D \rightarrow D$, be a continuous function, and suppose D is convex and compact. Then there exists x such that $f(x) = x$
  - Many other versions of fixed point theorems also available

Nash equilibria: Computation

- So far, we are not aware of efficient algorithms for finding fixed points [Hirsch, Papadimitriou, Vavasis '91]
  - There exist exponential time algorithms for finding approximate fixed points
- Can we design polynomial time algorithms for 2-player games?
  - After all, it seems to be only a special case of the general problem of finding fixed points
- For games with more players?

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- NP: The class of problems that can be verified in polynomial time
  - If I had a solution, I can verify if this solution is correct or not in polynomial time

- PPAD: The class of problems that we know that they have a solution, but this solution cannot be computed in polynomial time
  - Finding a MNE belongs in this class
  - Recall that according to Nash, there always exists a MNE

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- Recall that NE expresses stability as a solution concept
- Is this stable outcome good for the players?

Pareto Optimality

- We say that a state is Pareto Optimal if there is no other state that <u>all</u> the players have at least the same utility as before

Back to prisoner's dilemma

- Players = the two prisoners
- Strategies = {confess, silent}
- Possible states = {(confess, confess), (confess, silent), (silent, confess), (silent, silent)}
- Utilities given by the bi-matrix:

	confess	silent
confess	-3, -3	0, -5
silent	-5, 0	-1, -1

- Confessing is a best response to any strategy of the other player
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- (silent, silent) is a Pareto Optimal state

Social Welfare

- The **Social Welfare** (SW) of a state *s* is defined as the sum of the payoffs of the players at this state

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- Expresses how good the society feels with the outcome

Battle of the sexes

- Sports, sports has a Social Welfare of 9

		man	
		sports	movie
woman	sports	3, 6	1, 1
	movie	2, 2	6, 3

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- The price of stability is an *optimistic* measure: it considers the best equilibrium (with maximum social welfare)
- The price of anarchy is a *pessimistic* measure: it considers the worst equilibrium (with minimum social welfare)

Load balancing

	M_1	M_2
M_1	$4, 4$	$8, 4$
M_2	$4, 8$	$2, 2$

- There are three equilibrium states: (M_1, M_1), (M_1, M_2) and (M_2, M_1)
- (M_1, M_1) has social welfare 8, while (M_1, M_2) and (M_2, M_1) have social welfare 12 and are the optimal states

$$\text{PoS} = \frac{12}{12} = 1 \qquad \text{PoA} = \frac{3}{2}$$

Price of stability and price of anarchy

- $1 \leq PoS \leq PoA$
- The closer that we are in 1, the better (more efficient) the NE is

Exercise 1 (Dominant strategy equilibria)**.** Answer the following two questions.

1. Construct one 2-player zero-sum game that has a strict pure dominant strategy equilibrium and one that does not. It suffices to state the payoff matrices of the games and the dominant strategy equilibrium when it exists.

2. Can a game have more that one strict dominant strategy equilibrium? Explain your answer.

Exercise 2 (Dominant and Dominated Strategies)**.** Consider the following simple game. Alice and Julie have ten pairs of shoes to divide between them and each one has a strict preference over them, i.e. they are not indifferent between any two pairs of shoes. They come up with the following solution. They will both state their complete preferences over the shoes and then Alice will select her five favourite pairs first and Julie will get the rest.

1. What are the strategy spaces S_A and S_J of Alice and Julie respectively?
2. How many strictly dominant and how many weakly dominant strategies does Alice have?
3. How many strictly dominant and how many weakly dominant strategies does Julie have?
4. Does Alice have any strictly dominated strategies? What about Julie?
5. Does Alice have any weakly dominated strategies? What about Julie?

Exercise 3 (Second price auction)**.** Consider the following auction scenario. There is an item for sale and n interested bidders; the *valuation* of bidder i for the item is v_i, which represents how many pounds the bidder would be willing to spend on buying the item.

In a *second price auction*, the auctioneer asks the bidders to report their valuations v_i and then sells the item to the bidder with the highest bid at a price p equal to the second highest bid (break ties arbitrarily). All other agents (who do not receive the item) are charged 0. The payoff of any bidder i is 0 if she does not receive the item and $v_i - p$ if she does.

1. What is the (pure) strategy space S_i of each bidder i?
2. Show that for any bidder i, there is a strategy s_i that weakly dominates any strategy $s'_i > v_i$.
3. Show that for any bidder i, there is a strategy s_i that weakly dominates any strategy $s'_i < v_i$.
4. Does this game has a weak dominant strategy equilibrium? If not, explain your answer. If yes, state the equilibrium profile.