Algorithmic Game Theory
Solution concepts in games
Part II

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Based on slides by Vangelis Markakis and Alexandros Voudouris

Mixed equilibria

- **Mixed equilibrium:** A profile of *mixed* strategies such that each player maximizes its expected utility, given the strategies of the other players

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- Every pure equilibrium is also a mixed equilibrium
  - Every pure strategy can be seen as a probability distribution over all strategies that assigns probability 1 to this one pure strategy

Matching Pennies: mixed equilibria

		odd	
		heads	**tails**
even	**heads**	**1, -1**	**-1, 1**
	tails	**-1, 1**	**1, -1**

- Even player selects heads with probability x and tails with $1 - x$
- Odd player selects heads with probability y and tails with $1 - y$

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- $p(\text{heads, heads}) = xy$
- $p(\text{heads, tails}) = x(1 - y)$
- $p(\text{tails, heads}) = (1 - x)y$
- $p(\text{tails, tails}) = (1 - x)(1 - y)$

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		y	$1 - y$	

- $\mathbb{E}_p[u_{\mathrm{e}}]$
  $= xy \cdot 1 + x(1-y) \cdot (-1) + (1-x)y \cdot (-1) + (1-x)(1-y) \cdot 1$

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  $= 4xy - 2x \ - 2y + 1$
  $= \boldsymbol{x(4y-2) - 2y + 1}$

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- $\mathbb{E}_p[u_{\text{o}}]$
  $= xy \cdot (-1) + x(1-y) \cdot 1 + (1-x)y \cdot 1 + (1-x)(1-y) \cdot (-1)$
  $= \boldsymbol{y(2-4x) + 2x - 1}$

Matching Pennies: mixed equilibria

- $\mathbb{E}_p[u_\mathrm{e}] = x(4y - 2) - 2y + 1$
- $\mathbb{E}_p[u_\mathrm{o}] = y(2 - 4x) + 2x - 1$

- The expected utility of each player is a **linear function** in terms of her corresponding probability

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- **Negative:** the function is decreasing and the player aims to set a small value for the probability

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- The expected utility of each player is a **linear function** in terms of her corresponding probability
- To analyze how a player is going to act, we need to see whether the slope of the linear function is negative or positive

- **Negative:** the function is decreasing and the player aims to set a small value for the probability
- **Positive:** the function is increasing and the players aims to set a high value for the probability

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  - ⇨ **contradiction**

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- Following the same reasoning for the odd player, we can see that it must also be $\boldsymbol{x = 1/2}$

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- Following the same reasoning for the odd player, we can see that it must also be $\boldsymbol{x = 1/2}$
- For these values of x and y both slopes are equal to 0 and the linear functions are maximized
- The pair $(x, y) = (1/2, 1/2)$ corresponds to a mixed equilibrium, which is actually unique for this game

Unbalanced coordination

- Two players with two possible strategies A and B
- If both players select A, they get one point
- If both of them select B, they get two points
- If the select different strategies, they get zero points

		col player	
		A	**B**
row player	**A**	**1, 1**	**0, 0**
	B	**0, 0**	**2, 2**

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		A	**B**
row player	**A**	**1, 1**	**0, 0**
	B	**0, 0**	**2, 2**

- Easy to verify that (A, A) and (B, B) are pure equilibria
- Are there any other mixed equilibria?

Unbalanced coordination

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		A	B
row player	A	1, 1	0, 0
	B	0, 0	2, 2

- row player selects A with probability x and B with $1 - x$
- col player selects A with probability y and B with $1 - y$

- $p(\text{A}, \text{A}) = xy$
- $p(\text{A}, \text{B}) = x(1 - y)$
- $p(\text{B}, \text{A}) = (1 - x)y$
- $p(\text{B}, \text{B}) = (1 - x)(1 - y)$

Unbalanced coordination

		col player		
		A	**B**	
row player	**A**	**1, 1**	**0, 0**	x
	B	**0, 0**	**2, 2**	$1-x$
		y	$1-y$	

- $\mathbb{E}_p[u_\mathsf{r}]$
  $= xy \cdot 1 + x(1-y) \cdot 0 + (1-x)y \cdot 0 + (1-x)(1-y) \cdot 2$
  $= \boldsymbol{x(3y-2)+2-2y}$

- $\mathbb{E}_p[u_\mathsf{c}]$
  $= xy \cdot 1 + x(1-y) \cdot 0 + (1-x)y \cdot 0 + (1-x)(1-y) \cdot 2$
  $= \boldsymbol{y(3x-2)+2-2y}$

Unbalanced coordination

- $\mathbb{E}_p[u_{\mathsf{r}}] = x(3y-2) + 2 - 2y$
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Unbalanced coordination

- $\mathbb{E}_p[u_\mathsf{r}] = x(3y-2)+2-2y$
- $\mathbb{E}_p[u_\mathsf{c}] = y(3x-2)+2-2x$

- $\boldsymbol{y < 2/3}$
  ⇨ the slope $3y-2$ of $\mathbb{E}_p[u_\mathsf{r}]$ is **negative**

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  ⇨ row player sets $\boldsymbol{x = 0}$ to maximize $\mathbb{E}_p[u_\mathrm{r}]$

  ⇨ the slope $3x - 2 = -2$ of $\mathbb{E}_p[u_\mathrm{c}]$ is **negative**

  ⇨ the function $\mathbb{E}_p[u_\mathrm{c}]$ is **decreasing in $\boldsymbol{y}$**

  ⇨ column player sets $\boldsymbol{y = 0}$ to maximize $\mathbb{E}_p[u_\mathrm{c}]$

- $(x, y) = (0, 0)$ is a mixed equilibrium
- We already knew that: it corresponds to the pure equilibrium (A, A)

Unbalanced coordination

- $\mathbb{E}_p[u_{\mathsf{r}}] = x(3y - 2) + 2 - 2y$
- $\mathbb{E}_p[u_{\mathsf{c}}] = y(3x - 2) + 2 - 2x$

- $\boldsymbol{y > 2/3}$
  ⇨ the slope $3y - 2$ of $\mathbb{E}_p[u_{\mathsf{r}}]$ is **positive**

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  ⇨ the slope $3y-2$ of $\mathbb{E}_p[u_{\mathsf{r}}]$ is **positive**

  ⇨ the function $\mathbb{E}_p[u_{\mathsf{r}}]$ is **increasing in $\boldsymbol{x}$**

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  ⇨ the slope $3x-2=1$ of $\mathbb{E}_p[u_\mathsf{c}]$ is **positive**

Unbalanced coordination

- $\mathbb{E}_p[u_{\mathsf{r}}] = x(3y-2)+2-2y$
- $\mathbb{E}_p[u_{\mathsf{c}}] = y(3x-2)+2-2x$

- $\boldsymbol{y > 2/3}$
  ⇨ the slope $3y-2$ of $\mathbb{E}_p[u_{\mathsf{r}}]$ is **positive**
  ⇨ the function $\mathbb{E}_p[u_{\mathsf{r}}]$ is **increasing in $\boldsymbol{x}$**
  ⇨ row player sets $\boldsymbol{x = 1}$ to maximize $\mathbb{E}_p[u_{\mathsf{r}}]$
  ⇨ the slope $3x-2=1$ of $\mathbb{E}_p[u_{\mathsf{c}}]$ is **positive**
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- $(x, y) = (1, 1)$ is a mixed equilibrium corresponding to the pure equilibrium (B, B)

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- $\mathbb{E}_p[u_\mathsf{c}] = y(3x-2)+2-2x$

- For $\boldsymbol{x} < \mathbf{2/3}$ and $\boldsymbol{x} > \mathbf{2/3}$ we will reach to the same conclusion

- It remains to see what is going on for $\boldsymbol{x} = \mathbf{2/3}$ and $\boldsymbol{y} = \mathbf{2/3}$

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- For $x = 2/3$ the slope $3x - 2$ of $\mathbb{E}_p[u_{\mathsf{c}}]$ is zero and $\mathbb{E}_p[u_{\mathsf{c}}]$ is maximized by any choice of y, including $y = 2/3$

- $(x, y) = (2/3, 2/3)$ is a fully mixed equilibrium of the game

The Indifference Principle

- The same idea
- A different approach
- Find MNE in simple games (i.e. 2x2)

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- A different approach
- Find MNE in simple games (i.e. 2x2)
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The Indifference Principle

- These probabilities must not be 0 and 1 for their respective strategies
- If so this will be a PNE
  - Even if only one of the players always plays a strategy with probability 1, then there is a best response to that
  - This means that the other player will also play something with probability 1

The Indifference Principle

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- Compute the expected utility of each player for each of his strategies
- This player must be indifferent between the outcomes
  - If not then he would choose a strategy with certainty (probability 1)
- The expected utilities of a player must be equal
  - In that way we can compute the probabilities

Unbalanced coordination

- Two players with two possible strategies A and B
- If both players select A, they get one point
- If both of them select B, they get two points
- If the select different strategies, they get zero points

		col player	
		A	**B**
row player	**A**	**1, 1**	**0, 0**
	B	**0, 0**	**2, 2**

Unbalanced coordination

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- If both players select A, they get one point
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		col player: A	col player: B
row player	**A**	**1, 1**	**0, 0**
	B	**0, 0**	**2, 2**

- Easy to verify that (A, A) and (B, B) are pure equilibria
- Are there any other mixed equilibria? <u>Use the indifference principle!</u>

Unbalanced coordination

		col player	
		A	**B**
row player	**A**	**1, 1**	**0, 0**
	B	**0, 0**	**2, 2**

- row player selects A with probability x and B with $1 - x$
- col player selects A with probability y and B with $1 - y$

Unbalanced coordination

		col player		
		A	**B**	
row player	**A**	**1, 1**	**0, 0**	x
	B	**0, 0**	**2, 2**	$1-x$
		y	$1-y$	

- $\mathbb{E}_p[u_{\mathrm{rA}}] = y \cdot 1 + (1-y) \cdot 0 = \boldsymbol{y}$

- $\mathbb{E}_p[u_{\mathrm{rB}}] = y \cdot 0 + (1-y) \cdot 2 = \mathbf{2} - \mathbf{2}\boldsymbol{y}$

Unbalanced coordination

- $\mathbb{E}_p[u_{\mathsf{rA}}] = \mathbb{E}_p[rB]$
- $y = 2 - 2y$
- $y = 2/3$

Unbalanced coordination

		col player		
		A	**B**	
row player	**A**	**1, 1**	**0, 0**	x
	B	**0, 0**	**2, 2**	$1 - x$
		y	$1 - y$	

- $\mathbb{E}_p[u_{\mathrm{cA}}] = x \cdot 1 + (1 - x) \cdot 0 = \boldsymbol{x}$

- $\mathbb{E}_p[u_{\mathrm{cB}}] = x \cdot 0 + (1 - x) \cdot 2 = \mathbf{2} - \mathbf{2}\boldsymbol{x}$

Unbalanced coordination

- $\mathbb{E}_p\left[u_{\mathsf{cA}}\right] = \mathbb{E}_p[cB]$
- x$= 2 - 2x$
- x$= 2/3$

Unbalanced coordination

- The same result with both techniques!!!!

Multi-player games

Games with more than 2 players

- All the definitions we have seen can be generalized for multi-player games
  - Dominant strategies, Nash equilibria
- But: we can no longer have a representation with 2-dimensional arrays
- For n-player games we would need n-dimensional arrays (unless there is a more concise representation)

Definitions for n-player games

Definition: A game in normal form consists of

- A set of players $N = \{1, 2, \ldots, n\}$
- For every player i, a set of available pure strategies S^i
- For every player i, a utility function
  $u_i: S^1 \times \ldots \times S^n \rightarrow R$

- Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be a profile of mixed strategies for the players
- Each $\mathbf{p}_i$ is a probability distribution on S^i
- Expected utility of pl. i under $\mathbf{p}$ =

$$u_i(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \sum_{(s_1,\ldots,s_n) \in S^1 \times \cdots \times S^n} \mathbf{p}_1(s_1) \ldots \mathbf{p}_n(s_n) u_1(s_1, \ldots, s_n)$$

Notation

- Given a vector $s = (s_1, \ldots, s_n)$,
  we denote by s_{-i} the vector where we have removed the i-th coordinate:

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

- E.g., if $s = (3, 5, 7, 8)$, then
  - $s_{-3} = (3, 5, 8)$
  - $s_{-1} = (5, 7, 8)$
- We can write a strategy profile s as $s = (s_i, s_{-i})$

Definitions for n-player games

- A strategy $\mathbf{p}_i$ of pl. i is *dominant* if

  $$u_i\,(\mathbf{p}_i, \mathbf{p}_{-i}) \geq u_i\,(e^j, \mathbf{p}_{-i})$$

  for every pure strategy e^j of pl. i, and every profile $\mathbf{p}_{-i}$ of the other players
- Replace $\geq$ with $>$ for strictly dominant

- A profile $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is a Nash equilibrium if for every player i and every pure strategy e^j of pl. i, we have

  $$u_i(\mathbf{p}) \geq u_i(e^j, \mathbf{p}_{-i})$$

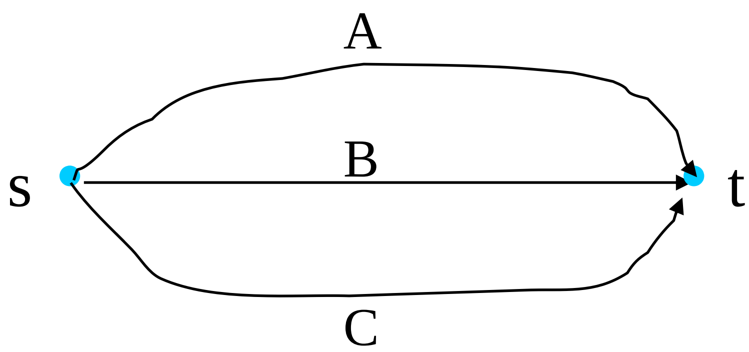
  - As in 2-player games, it suffices to check only deviations to pure strategies

Nash equilibria in multi-player games

At a first glance:

- Even finding pure Nash equilibria looks already more difficult than in the 2-player case
- We can try with brute force all possible profiles
- Suppose we have n players, and each of them has m strategies: $|S^i| = m$
- There are m^n pure strategy profiles!
- However, in some cases, we can exploit symmetry or other properties to reduce our search space

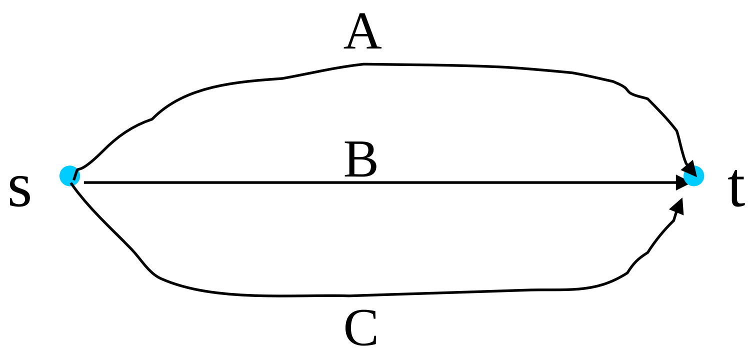
Example: Congestion games



A simple example of a congestion game:

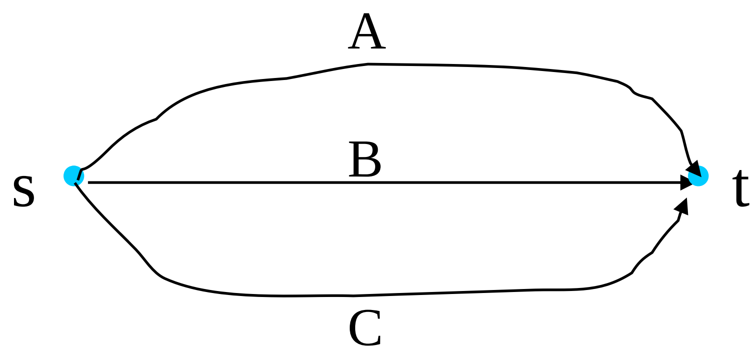
- A set of network users wants to move from s to t
- 3 possible routes, A, B, C
- Time delay in a route: depends on the number of users who have chosen this route
- $d_A(x) = 5x$, $d_B(x) = 7.5x$, $d_C(x) = 10x$,

Example: Congestion games



- Suppose we have n = 5 players
- For each player i, S^i = {A, B, C}
- Number of possible pure strategy profiles: $3^5 = 243$
- Utility function of a player: should increase when delay decreases (e.g., we can define it as u = – delay)
- At profile s = (A, C, A, B, A)
  - $u_1(s) = -15$, $u_2(s) = -10$, $u_3(s) = -15$, $u_4(s) = -7.5$, $u_5(s) = -15$

Example: Congestion games



- There is no need to examine all 243 possible profiles to find a pure equilibrium
- Exploiting symmetry:
  - In every route, the delay does not depend on who chose the route but only how many did so
- We can also exploit further properties
  - E.g. There can be no equilibrium where one of the routes is not used by some player

Homework: Find the pure Nash equilibria of this game (if there are any)

Game simplifications:
Strict and weak domination

Strictly dominated strategies

- In Prisoner's dilemma, we saw that strategy C is dominant
- Strategy D is "dominated" by C
- Definition: A (pure or mixed) strategy $\mathbf{p}_i$ of pl. i strictly dominates some other strategy $\mathbf{p}'$ if for every profile $\mathbf{p}_{-i}$ of the other players, it holds that

$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) > u_i(\mathbf{p}', \mathbf{p}_{-i})$$

- Strategy $\mathbf{p}'$ will be called strictly dominated
- Observation: it suffices to consider only profiles $\mathbf{p}_{-i}$ with pure strategies

Strictly dominated strategies

- Strictly dominated strategies cannot be used in any Nash equilibrium

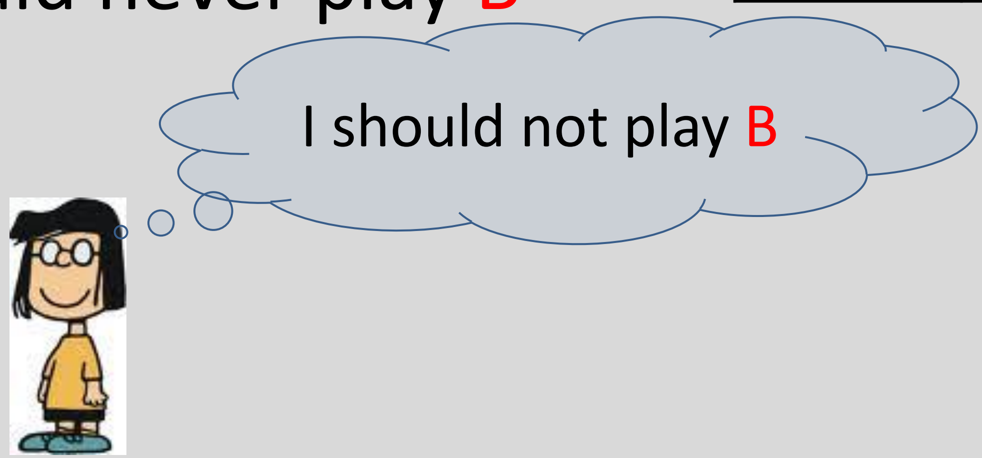
- Hence, we can remove them and reduce the size of the game

- In some cases, this results in much simpler games to analyze

Iterated Elimination of Strictly Dominated Actions

- Action B of player 1 is dominated by T or C
- None of the actions of player 2 is dominated
- If player 1 is rational, she would never play B

	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	~~2, 0~~	~~2, 0~~	~~2, 6~~



Iterated Elimination of Strictly Dominated Actions

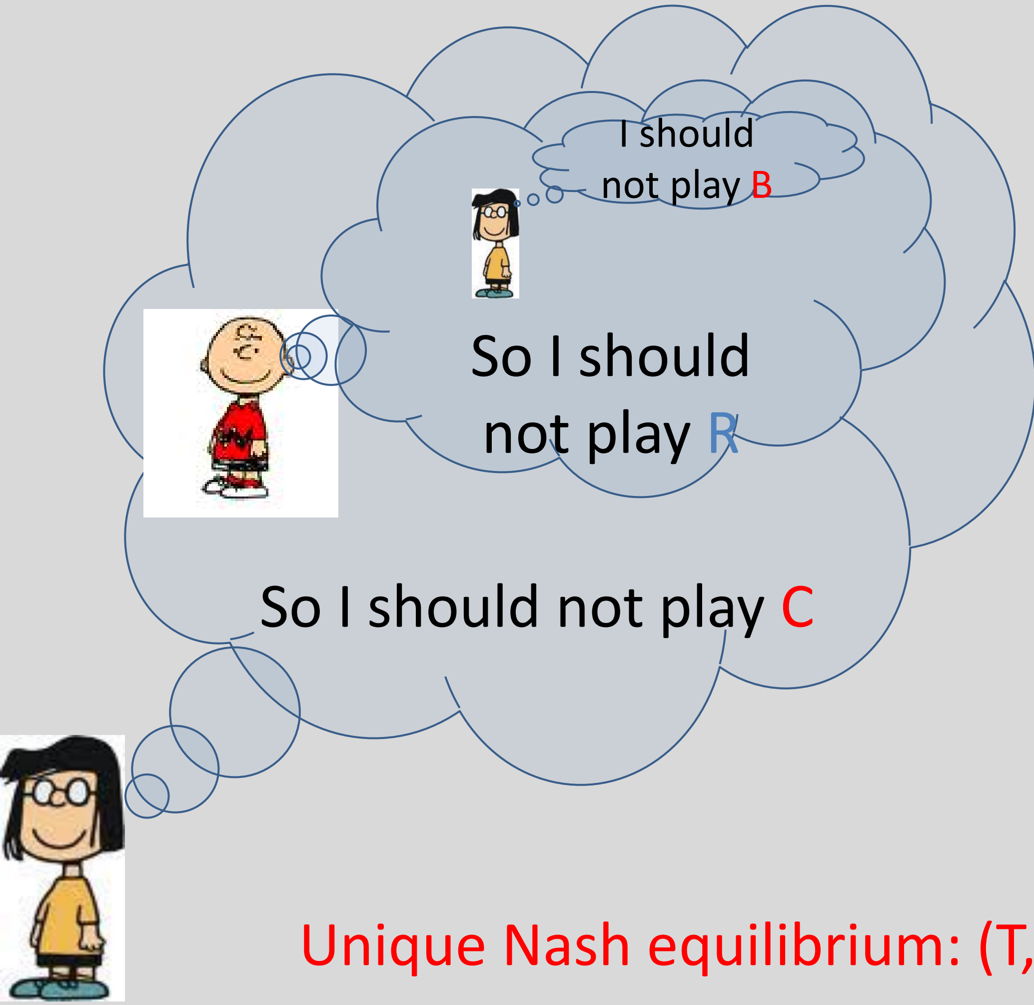
- If player 2 knows player 1 is rational, he can assume player 1 does not play B
  - then player 2 should not play R

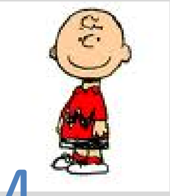
	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	~~2, 0~~	~~2, 0~~	~~2, 6~~

I should not play B

So I should not play R

Iterated Elimination of Strictly Dominated Actions





	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	2, 0	2, 0	2, 6

Unique Nash equilibrium: (T, L)

Strict domination by mixed strategies

- **<u>Attention:</u>** It is possible that some strategy is not strictly dominated by a pure strategy but it is dominated by a mixed strategy
- Strategy B of pl. 1 is not strictly dominated neither by T nor by C
- But, it is strictly dominated by the mixed strategy (1/2, 1/2, 0), i.e., 0.5T + 0.5C:
  - Proof: Consider some arbitrary strategy of pl. 2 $\mathbf{q} = (q_1, 1-q_1)$
  - $u_1(B, \mathbf{q}) = 2$
  - $u_1((1/2, 1/2, 0), \mathbf{q}) = 1/2 \times q_1 \times 5 + 1/2 \times (1-q_1) \times 5 = 2.5 > 2$

	L	R
T	5, 5	0, 0
C	0, 0	5, 5
B	2, 0	2, 0

Strict domination by mixed strategies

- Consider a 2-player game with $S^1 = \{s_1, s_2, \ldots, s_n\}$, $S^2 = \{t_1, t_2, \ldots, t_m\}$
- How can we check if the pure strategy s_i of pl. 1 is strictly dominated by some other (possibly mixed) strategy?
- We have to check if there exist probabilities $p_1, \ldots, p_n$ such that
  - For every $t_j \in S^2$ (for every column), $u_1(s_i, t_j) < p_1 u_1(s_1, t_j) + \ldots + p_n u_1(s_n, t_j)$
  - also, $p_1 + p_2 + \ldots + p_n = 1$, $p_i \geq 0$ for $i = 1, \ldots, n$
- System with linear inequalities, it has a solution iff s_i is strictly dominated

Iterated Elimination of Strictly Dominated Actions

- Given: an n-player game
  - pick a player i that has a strictly dominated pure strategy (dominated either by a pure or mixed strategy)
  - Remove one of the strictly dominated strategies of pl. i
  - repeat until no player has a strictly dominated pure strategy

- <u>Facts</u>:
  - the set of surviving actions is independent of the elimination order, i.e., which player was picked at each step
  - Iterated elimination of strictly dominated actions cannot destroy Nash equilibria

Weakly dominated strategies

- Definition: A (pure or mixed) strategy $\mathbf{p}_i$ of pl. i weakly dominates some other strategy $\mathbf{p}'$ if for every profile $\mathbf{p}_{-i}$ of the other players, it holds that

$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \geq u_i(\mathbf{p}', \mathbf{p}_{-i})$$

and for at least one profile $\mathbf{p}_{-i}$ we have

$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) > u_i(\mathbf{p}', \mathbf{p}_{-i})$$

- Strategy $\mathbf{p}'$ will be called weakly dominated

Weakly dominated strategies

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

	L	R
T	2, 2	3, 0
B	0, 3	3, 3

- When we remove weakly dominated strategies, we may lose some Nash equilibria
- In the above games:
  - Strategy T weakly dominates B
  - Strategy L weakly dominates R
  - but (B, R) is an equilibrium
- Observation: In the 2^{nd} game, we even have a better value for both players when they choose weakly dominated strategies

Iterated Elimination of Weakly Dominated Actions and Nash Equilibria

- The elimination order matters in iterated deletion of weakly dominated strategies
- Each order may eliminate a different subset of Nash equilibria
- Can we lose all equilibria of the original game?

- Theorem: For every game where each player has a finite strategy space, there is always at least one equilibrium that survives iterated elimination of weakly dominated strategies
  - thus: if we care for finding just one Nash equilibrium, no need to worry about elimination order

Exercise

	t_1	t_2
s_1	3, 2	2, 2
s_2	1, 1	0, 0
s_3	0, 0	1, 1

Execute all the possible ways of doing iterated elimination of weakly dominated strategies. Do we lose equilibria with this process?