Social Networks and Online Markets

Homework 2

Due: 7/6/2020, 23:59

Instructions

You must hand in the homeworks electronically and before the due date and time.

Handing in: You must hand in the homeworks by the due date and time by an email to **both** `aris@diag.uniroma1.it` and `schwiegelshohn@diag.uniroma1.it` that will contain as attachment **(not links to some file-uploading server!)** a .zip or .pdf file with your answers.
After you submit, you will receive an acknowledgement email that your homework has been received. If you have not received an acknowledgement email within 2 days after the deadline then contact Aris and Chris.

The solutions for the theoretical exercises must contain your answers either typed up or hand written clearly and scanned. They should not require more than half page per question.

For information about collaboration, and about being late check the web page.

Problem 1. We are given a matrix $A \in \mathbb{R}^{n \times d}$. Suppose we have two collections of pairwise orthogonal unit vectors $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, such that $span(\{v_1, \ldots, v_k\}) = span(\{w_1, \ldots, w_k\})$. Show that

$$\sum_{i=1}^{k} \|Av_i\|^2 = \sum_{i=1}^{k} \|Aw_i\|^2.$$

Problem 2. We are given a graph G with adjacency matrix A, degree matrix D and Laplacian matrix $L = D - A$. Find an example for G such that the eigenvector associated to the smallest eigenvalue of L is not an eigenvector of A.

Problem 3. Recall that the sparsest cut is a partition of the nodes into two sets S and $V \setminus S$ such that $\frac{|E \cap (S \times (V \setminus S))|}{\min(|S|, |V \setminus S|)}$ is minimized. The densest subgraph D is the subset $D \subset V$ maximizing $\frac{|E \cap (D \times D)|}{|D|}$. Find a graph such that $D \notin \{S, V \setminus S\}$.

Problem 4. Give an example for an asymmetric space with n points such that $d(a,b) = 1$ and $d(b,a) = n - 1$, for some pair of points a and b.

Problem 5. Give an example for a graph, in which the adjacency matrix has no negative eigenvalues. What is an alternative name for the resulting graph?

Problem 6. (**Bonus**) Segment the image found here: `https://www.skuola.net/news_foto/2020/Diodato.jpg` using the spectrum of the graph. If you missed the lecture on applications of sparsest cut, you'll find the information in the paper "Normalized Cuts and Image Segmentation" by Shi and Malik.