Chapter 8

NP-complete problems

8.1 Search problems

Over the past seven chapters we have developed algorithms for finding shortest paths and minimum spanning trees in graphs, matchings in bipartite graphs, maximum increasing subsequences, maximum flows in networks, and so on. All these algorithms are *efficient*, because in each case their time requirement grows as a polynomial function (such as n, n^2, or n^3) of the size of the input.

To better appreciate such efficient algorithms, consider the alternative: In all these problems we are searching for a solution (path, tree, matching, etc.) from among an *exponential* population of possibilities. Indeed, n boys can be matched with n girls in $n!$ different ways, a graph with n vertices has n^{n-2} spanning trees, and a typical graph has an exponential number of paths from s to t. All these problems could in principle be solved in exponential time by checking through all candidate solutions, one by one. But an algorithm whose running time is 2^n, or worse, is all but useless in practice (see the next box). The quest for efficient algorithms is about finding clever ways to bypass this process of exhaustive search, using clues from the input in order to dramatically narrow down the search space.

So far in this book we have seen the most brilliant successes of this quest, algorithmic techniques that defeat the specter of exponentiality: greedy algorithms, dynamic programming, linear programming (while divide-and-conquer typically yields faster algorithms for problems we can already solve in polynomial time). Now the time has come to meet the quest's most embarrassing and persistent failures. We shall see some other "search problems," in which again we are seeking a solution with particular properties among an exponential chaos of alternatives. But for these new problems no shortcut seems possible. The fastest algorithms we know for them are all exponential—not substantially better than an exhaustive search. We now introduce some important examples.

The story of Sissa and Moore

According to the legend, the game of chess was invented by the Brahmin Sissa to amuse and teach his king. Asked by the grateful monarch what he wanted in return, the wise man requested that the king place one grain of rice in the first square of the chessboard, two in the second, four in the third, and so on, doubling the amount of rice up to the 64th square. The king agreed on the spot, and as a result he was the first person to learn the valuable—-albeit humbling—lesson of *exponential growth*. Sissa's request amounted to $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$ grains of rice, enough rice to pave all of India several times over!

All over nature, from colonies of bacteria to cells in a fetus, we see systems that grow exponentially—for a while. In 1798, the British philosopher T. Robert Malthus published an essay in which he predicted that the exponential growth (he called it "geometric growth") of the human population would soon deplete linearly growing resources, an argument that influenced Charles Darwin deeply. Malthus knew the fundamental fact that an exponential sooner or later takes over any polynomial.

In 1965, computer chip pioneer Gordon E. Moore noticed that transistor density in chips had doubled every year in the early 1960s, and he predicted that this trend would continue. This prediction, moderated to a doubling every 18 months and extended to computer speed, is known as *Moore's law*. It has held remarkably well for 40 years. And these are the two root causes of the explosion of information technology in the past decades: *Moore's law and efficient algorithms*.

It would appear that Moore's law provides a disincentive for developing polynomial algorithms. After all, if an algorithm is exponential, why not wait it out until Moore's law makes it feasible? But in reality the exact opposite happens: Moore's law is a huge incentive for developing efficient algorithms, because such algorithms are needed in order to take advantage of the exponential increase in computer speed.

Here is why. If, for example, an $O(2^n)$ algorithm for Boolean satisfiability (SAT) were given an hour to run, it would have solved instances with 25 variables back in 1975, 31 variables on the faster computers available in 1985, 38 variables in 1995, and about 45 variables with today's machines. Quite a bit of progress—except that each extra variable requires a year and a half's wait, while the appetite of applications (many of which are, ironically, related to computer design) grows much faster. In contrast, the size of the instances solved by an $O(n)$ or $O(n \log n)$ algorithm would be *multiplied by a factor of about 100* each decade. In the case of an $O(n^2)$ algorithm, the instance size solvable in a fixed time would be multiplied by about 10 each decade. Even an $O(n^6)$ algorithm, polynomial yet unappetizing, would more than double the size of the instances solved each decade. When it comes to the growth of the size of problems we can attack with an algorithm, we have a reversal: exponential algorithms make polynomially slow progress, while polynomial algorithms advance exponentially fast! For Moore's law to be reflected in the world we *need* efficient algorithms.

As Sissa and Malthus knew very well, exponential expansion cannot be sustained indefinitely in our finite world. Bacterial colonies run out of food; chips hit the atomic scale. Moore's law will stop doubling the speed of our computers within a decade or two. And then progress will depend on algorithmic ingenuity—or otherwise perhaps on novel ideas such as *quantum computation*, explored in Chapter 10.

Satisfiability

SATISFIABILITY, or SAT (recall Exercise 3.28 and Section 5.3), is a problem of great practical importance, with applications ranging from chip testing and computer design to image analysis and software engineering. It is also a canonical hard problem. Here's what an instance of SAT looks like:

$$(x \vee y \vee z)\ (x \vee \overline{y})\ (y \vee \overline{z})\ (z \vee \overline{x})\ (\overline{x} \vee \overline{y} \vee \overline{z}).$$

This is a *Boolean formula in conjunctive normal form (CNF)*. It is a collection of *clauses* (the parentheses), each consisting of the disjunction (logical *or*, denoted $\vee$) of several *literals*, where a literal is either a Boolean variable (such as x) or the negation of one (such as $\overline{x}$). A *satisfying truth assignment* is an assignment of `false` or `true` to each variable so that every clause contains a literal whose value is `true`. The SAT problem is the following: given a Boolean formula in conjunctive normal form, either find a satisfying truth assignment or else report that none exists.

In the instance shown previously, setting all variables to `true`, for example, satisfies every clause except the last. Is there a truth assignment that satisfies *all* clauses?

With a little thought, it is not hard to argue that in this particular case no such truth assignment exists. (*Hint*: The three middle clauses constrain all three variables to have the same value.) But how do we decide this in general? Of course, we can always search through all truth assignments, one by one, but for formulas with n variables, the number of possible assignments is exponential, 2^n.

SAT is a typical *search problem*. We are given an *instance* I (that is, some input data specifying the problem at hand, in this case a Boolean formula in conjunctive normal form), and we are asked to find a *solution* S (an object that meets a particular specification, in this case an assignment that satisfies each clause). If no such solution exists, we must say so.

More specifically, a search problem must have the property that any proposed solution S to an instance I can be *quickly checked* for correctness. What does this entail? For one thing, S must at least be concise (quick to read), with length polynomially bounded by that of I. This is clearly true in the case of SAT, for which S is an assignment to the variables. To formalize the notion of quick checking, we will say that there is a polynomial-time algorithm that takes as input I and S and decides whether or not S is a solution of I. For SAT, this is easy as it just involves checking whether the assignment specified by S indeed satisfies every clause in I.

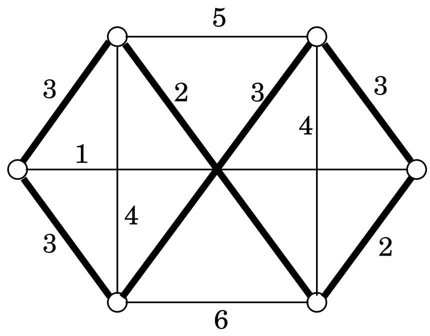
Later in this chapter it will be useful to shift our vantage point and to think of this efficient algorithm for checking proposed solutions as *defining* the search problem. Thus:

> A *search problem* is specified by an algorithm $\mathcal{C}$ that takes two inputs, an instance I and a proposed solution S, and runs in time polynomial in $|I|$. We say S is a solution to I if and only if $\mathcal{C}(I, S) = \texttt{true}$.

Given the importance of the SAT search problem, researchers over the past 50 years have tried hard to find efficient ways to solve it, but without success. The fastest algorithms we have are still exponential on their worst-case inputs.

Yet, interestingly, there are two natural variants of SAT for which we do have good algorithms. If all clauses contain at most one positive literal, then the Boolean formula is called

Figure 8.1 The optimal traveling salesman tour, shown in bold, has length 18.



a *Horn formula*, and a satisfying truth assignment, if one exists, can be found by the greedy algorithm of Section 5.3. Alternatively, if all clauses have only *two* literals, then graph theory comes into play, and SAT can be solved in linear time by finding the strongly connected components of a particular graph constructed from the instance (recall Exercise 3.28). In fact, in Chapter 9, we'll see a different polynomial algorithm for this same special case, which is called 2SAT.

On the other hand, if we are just a little more permissive and allow clauses to contain *three* literals, then the resulting problem, known as 3SAT (an example of which we saw earlier), once again becomes hard to solve!

The traveling salesman problem

In the traveling salesman problem (TSP) we are given n vertices $1, \ldots, n$ and all $n(n-1)/2$ distances between them, as well as a *budget* b. We are asked to find a *tour*, a cycle that passes through every vertex exactly once, of total cost b or less—or to report that no such tour exists. That is, we seek a permutation $\tau(1), \ldots, \tau(n)$ of the vertices such that when they are toured in this order, the total distance covered is at most b:

$$d_{\tau(1),\tau(2)} + d_{\tau(2),\tau(3)} + \cdots + d_{\tau(n),\tau(1)} \leq b.$$

See Figure 8.1 for an example (only some of the distances are shown; assume the rest are very large).

Notice how we have defined the TSP as a *search problem*: given an instance, find a tour within the budget (or report that none exists). But why are we expressing the traveling salesman problem in this way, when in reality it is an *optimization problem*, in which the *shortest* possible tour is sought? Why dress it up as something else?

For a good reason. Our plan in this chapter is to compare and relate problems. The framework of search problems is helpful in this regard, because it encompasses optimization problems like the TSP in addition to true search problems like SAT.

Turning an optimization problem into a search problem does not change its difficulty at all, because the two versions *reduce to one another*. Any algorithm that solves the optimization

TSP also readily solves the search problem: find the optimum tour and if it is within budget, return it; if not, there is no solution.

Conversely, an algorithm for the search problem can also be used to solve the optimization problem. To see why, first suppose that we somehow knew the *cost* of the optimum tour; then we could find this tour by calling the algorithm for the search problem, using the optimum cost as the budget. Fine, but how do we find the optimum cost? Easy: By binary search! (See Exercise 8.1.)

Incidentally, there is a subtlety here: Why do we have to introduce a budget? Isn't any optimization problem also a search problem in the sense that we are searching for a solution that has the property of being optimal? The catch is that the solution to a search problem should be easy to recognize, or as we put it earlier, polynomial-time checkable. Given a potential solution to the TSP, it is easy to check the properties "is a tour" (just check that each vertex is visited exactly once) and "has total length $\leq b$." But how could one check the property "is optimal"?

As with SAT, there are no known polynomial-time algorithms for the TSP, despite much effort by researchers over nearly a century. Of course, there is an exponential algorithm for solving it, by trying all $(n-1)!$ tours, and in Section 6.6 we saw a faster, yet still exponential, dynamic programming algorithm.

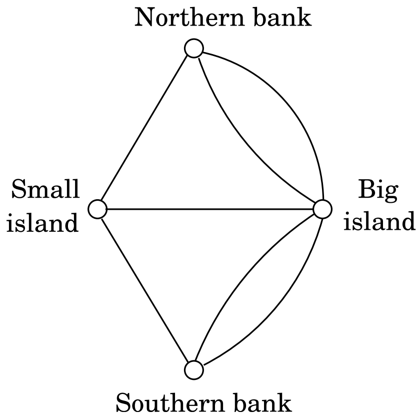
The minimum spanning tree (MST) problem, for which we *do* have efficient algorithms, provides a stark contrast here. To phrase it as a search problem, we are again given a distance matrix and a bound b, and are asked to find a tree T with total weight $\sum_{(i,j)\in T} d_{ij} \leq b$. The TSP can be thought of as a tough cousin of the MST problem, in which the tree is not allowed to branch and is therefore a path.[1] This extra restriction on the structure of the tree results in a much harder problem.

Euler and Rudrata

In the summer of 1735 Leonhard Euler (pronounced "Oiler"), the famous Swiss mathematician, was walking the bridges of the East Prussian town of Königsberg. After a while, he noticed in frustration that, no matter where he started his walk, no matter how cleverly he continued, it was impossible to cross each bridge exactly once. And from this silly ambition, the field of graph theory was born.

Euler identified at once the roots of the park's deficiency. First, you turn the map of the park into a graph whose vertices are the four land masses (two islands, two banks) and whose edges are the seven bridges:

[1]Actually the TSP demands a cycle, but one can define an alternative version that seeks a path, and it is not hard to see that this is just as hard as the TSP itself.



This graph has multiple edges between two vertices—a feature we have not been allowing so far in this book, but one that is meaningful for this particular problem, since each bridge must be accounted for separately. We are looking for a path that goes through each edge exactly once (the path is allowed to repeat vertices). In other words, we are asking this question: *When can a graph be drawn without lifting the pencil from the paper?*

The answer discovered by Euler is simple, elegant, and intuitive: *If and only if (a) the graph is connected and (b) every vertex, with the possible exception of two vertices (the start and final vertices of the walk), has even degree* (Exercise 3.26). This is why Königsberg's park was impossible to traverse: all four vertices have odd degree.

To put it in terms of our present concerns, let us define a search problem called EULER PATH: Given a graph, find a path that contains each edge exactly once. It follows from Euler's observation, and a little more thinking, that this search problem can be solved in polynomial time.

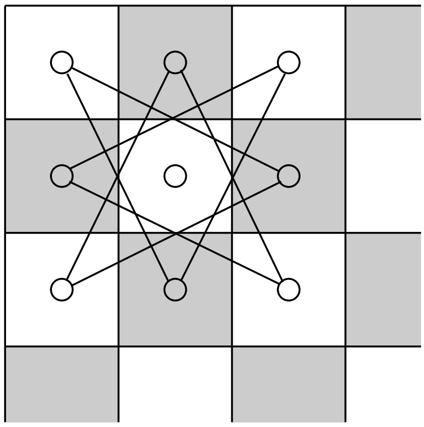
Almost a millennium before Euler's fateful summer in East Prussia, a Kashmiri poet named Rudrata had asked this question: Can one visit all the squares of the chessboard, without repeating any square, in one long walk that ends at the starting square and at each step makes a legal knight move? This is again a graph problem: the graph now has 64 vertices, and two squares are joined by an edge if a knight can go from one to the other in a single move (that is, if their coordinates differ by 2 in one dimension and by 1 in the other). See Figure 8.2 for the portion of the graph corresponding to the upper left corner of the board. Can you find a knight's tour on your chessboard?

This is a different kind of search problem in graphs: we want a cycle that goes through all *vertices* (as opposed to all edges in Euler's problem), without repeating any vertex. And there is no reason to stick to chessboards; this question can be asked of any graph. Let us define the RUDRATA CYCLE search problem to be the following: given a graph, find a cycle that visits each vertex exactly once—or report that no such cycle exists.[2] This problem is ominously reminiscent of the TSP, and indeed no polynomial algorithm is known for it.

There are two differences between the definitions of the Euler and Rudrata problems. The first is that Euler's problem visits all *edges* while Rudrata's visits all *vertices*. But there is

[2]In the literature this problem is known as the *Hamilton cycle* problem, after the great Irish mathematician who rediscovered it in the 19th century.

Figure 8.2 Knight's moves on a corner of a chessboard.



also the issue that one of them demands a path while the other requires a cycle. Which of these differences accounts for the huge disparity in computational complexity between the two problems? It must be the first, because the second difference can be shown to be purely cosmetic. Indeed, define the RUDRATA PATH problem to be just like RUDRATA CYCLE, except that the goal is now to find a *path* that goes through each vertex exactly once. As we will soon see, there is a precise equivalence between the two versions of the Rudrata problem.

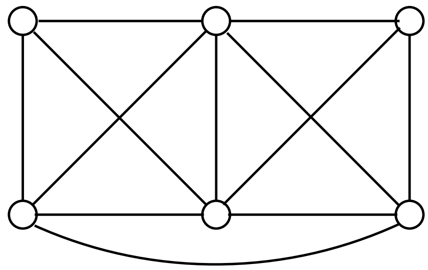
Cuts and bisections

A *cut* is a set of edges whose removal leaves a graph disconnected. It is often of interest to find small cuts, and the MINIMUM CUT problem is, given a graph and a budget b, to find a cut with at most b edges. For example, the smallest cut in Figure 8.3 is of size 3. This problem can be solved in polynomial time by $n-1$ max-flow computations: give each edge a capacity of 1, and find the maximum flow between some fixed node and every single other node. The smallest such flow will correspond (via the max-flow min-cut theorem) to the smallest cut. Can you see why? We've also seen a very different, randomized algorithm for this problem (page 150).

In many graphs, such as the one in Figure 8.3, the smallest cut leaves just a singleton vertex on one side—it consists of all edges adjacent to this vertex. Far more interesting are small cuts that partition the vertices of the graph into nearly equal-sized sets. More precisely, the BALANCED CUT problem is this: given a graph with n vertices and a budget b, partition the vertices into two sets S and T such that $|S|, |T| \geq n/3$ and such that there are at most b edges between S and T. Another hard problem.

Balanced cuts arise in a variety of important applications, such as *clustering*. Consider for example the problem of segmenting an image into its constituent components (say, an elephant standing in a grassy plain with a clear blue sky above). A good way of doing this is to create a graph with a node for each pixel of the image and to put an edge between nodes whose corresponding pixels are spatially close together and are also similar in color. A single

Figure 8.3 What is the smallest cut in this graph?



object in the image (like the elephant, say) then corresponds to a set of highly connected vertices in the graph. A balanced cut is therefore likely to divide the pixels into two clusters without breaking apart any of the primary constituents of the image. The first cut might, for instance, separate the elephant on the one hand from the sky and from grass on the other. A further cut would then be needed to separate the sky from the grass.

Integer linear programming

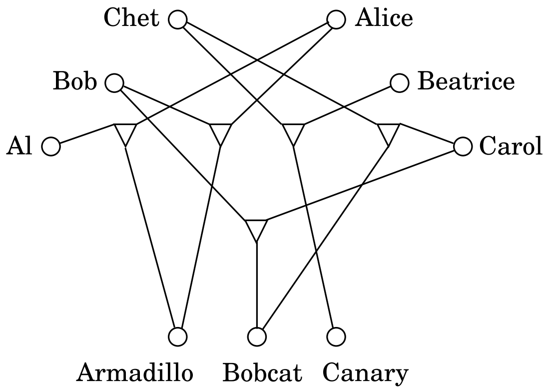
Even though the simplex algorithm is not polynomial time, we mentioned in Chapter 7 that there *is* a different, polynomial algorithm for linear programming. Therefore, linear programming is efficiently solvable both in practice and in theory. But the situation changes completely if, in addition to specifying a linear objective function and linear inequalities, we also constrain the solution (the values for the variables) to be *integer*. This latter problem is called INTEGER LINEAR PROGRAMMING (ILP). Let's see how we might formulate it as a search problem. We are given a set of linear inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an m-vector; an objective function specified by an n-vector $\mathbf{c}$; and finally, a *goal* g (the counterpart of a budget in maximization problems). We want to find a nonnegative *integer* n-vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c} \cdot \mathbf{x} \geq g$.

But there is a redundancy here: the last constraint $\mathbf{c} \cdot \mathbf{x} \geq g$ is itself a linear inequality and can be absorbed into $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. So, we define ILP to be following search problem: given $\mathbf{A}$ and $\mathbf{b}$, find a nonnegative integer vector $\mathbf{x}$ satisfying the inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, or report that none exists. Despite the many crucial applications of this problem, and intense interest by researchers, no efficient algorithm is known for it.

There is a particularly clean special case of ILP that is very hard in and of itself: the goal is to find a vector $\mathbf{x}$ of 0's and 1's satisfying $\mathbf{A}\mathbf{x} = \mathbf{1}$, where $\mathbf{A}$ is an $m \times n$ matrix with $0-1$ entries and $\mathbf{1}$ is the m-vector of all 1's. It should be apparent from the reductions in Section 7.1.4 that this is indeed a special case of ILP. We call it ZERO-ONE EQUATIONS (ZOE).

We have now introduced a number of important search problems, some of which are familiar from earlier chapters and for which there are efficient algorithms, and others which are different in small but crucial ways that make them very hard computational problems. To

Figure 8.4 A more elaborate matchmaking scenario. Each triple is shown as a triangular-shaped node joining boy, girl, and pet.



complete our story we will introduce a few more hard problems, which will play a role later in the chapter, when we relate the computational difficulty of all these problems. The reader is invited to skip ahead to Section 8.2 and then return to the definitions of these problems as required.

Three-dimensional matching

Recall the BIPARTITE MATCHING problem: given a bipartite graph with n nodes on each side (the *boys* and the *girls*), find a set of n disjoint edges, or decide that no such set exists. In Section 7.3, we saw how to efficiently solve this problem by a reduction to maximum flow. However, there is an interesting generalization, called 3D MATCHING, for which no polynomial algorithm is known. In this new setting, there are n boys and n girls, but also n *pets*, and the compatibilities among them are specified by a set of *triples*, each containing a boy, a girl, and a pet. Intuitively, a triple (b, g, p) means that boy b, girl g, and pet p get along well together. We want to find n disjoint triples and thereby create n harmonious households.

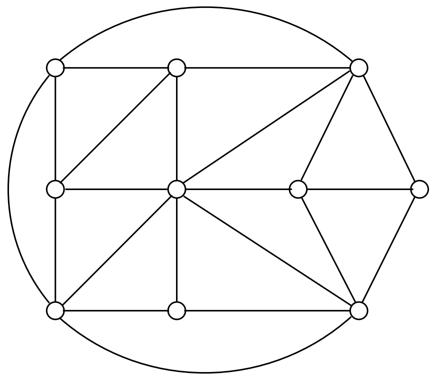
Can you spot a solution in Figure 8.4?

Independent set, vertex cover, and clique

In the INDEPENDENT SET problem (recall Section 6.7) we are given a graph and an integer g, and the aim is to find g vertices that are independent, that is, no two of which have an edge between them. Can you find an independent set of three vertices in Figure 8.5? How about four vertices? We saw in Section 6.7 that this problem can be solved efficiently on trees, but for general graphs no polynomial algorithm is known.

There are many other search problems about graphs. In VERTEX COVER, for example, the input is a graph and a budget b, and the idea is to find b vertices that cover (touch) every edge. Can you cover all edges of Figure 8.5 with seven vertices? With six? (And do you see the

Figure 8.5 What is the size of the largest independent set in this graph?



intimate connection to the INDEPENDENT SET problem?)

VERTEX COVER is a special case of SET COVER, which we encountered in Chapter 5. In that problem, we are given a set E and several subsets of it, $S_1, \ldots, S_m$, along with a budget b. We are asked to select b of these subsets so that their union is E. VERTEX COVER is the special case in which E consists of the edges of a graph, and there is a subset S_i for each vertex, containing the edges adjacent to that vertex. Can you see why 3D MATCHING is also a special case of SET COVER?

And finally there is the CLIQUE problem: given a graph and a goal g, find a set of g vertices such that all possible edges between them are present. What is the largest clique in Figure 8.5?

Longest path

We know the shortest-path problem can be solved very efficiently, but how about the LONGEST PATH problem? Here we are given a graph G with nonnegative edge weights and two distinguished vertices s and t, along with a goal g. We are asked to find a path from s to t with total weight at least g. Naturally, to avoid trivial solutions we require that the path be *simple*, containing no repeated vertices.

No efficient algorithm is known for this problem (which sometimes also goes by the name of TAXICAB RIP-OFF).

Knapsack and subset sum

Recall the KNAPSACK problem (Section 6.4): we are given integer weights $w_1, \ldots, w_n$ and integer values $v_1, \ldots, v_n$ for n items. We are also given a weight capacity W and a goal g (the former is present in the original optimization problem, the latter is added to make it a search problem). We seek a set of items whose total weight is at most W and whose total value is at least g. As always, if no such set exists, we should say so.

In Section 6.4, we developed a dynamic programming scheme for KNAPSACK with running time $O(nW)$, which we noted is exponential in the input size, since it involves W rather than $\log W$. And we have the usual exhaustive algorithm as well, which looks at all subsets of items—all 2^n of them. Is there a polynomial algorithm for KNAPSACK? Nobody knows of one.

But suppose that we are interested in the variant of the knapsack problem in which the integers are coded in *unary*—for instance, by writing $IIIIIIIIIIII$ for 12. This is admittedly an exponentially wasteful way to represent integers, but it does define a legitimate problem, which we could call UNARY KNAPSACK. It follows from our discussion that this somewhat artificial problem does have a polynomial algorithm.

A different variation: suppose now that each item's value is equal to its weight (all given in binary), and to top it off, the goal g is the same as the capacity W. (To adapt the silly break-in story whereby we first introduced the knapsack problem, the items are all gold nuggets, and the burglar wants to fill his knapsack to the hilt.) This special case is tantamount to finding a subset of a given set of integers that adds up to exactly W. Since it is a special case of KNAPSACK, it cannot be any harder. But could it be polynomial? As it turns out, this problem, called SUBSET SUM, is also very hard.

At this point one could ask: If SUBSET SUM is a special case that happens to be as hard as the general KNAPSACK problem, why are we interested in it? The reason is *simplicity*. In the complicated calculus of reductions between search problems that we shall develop in this chapter, conceptually simple problems like SUBSET SUM and 3SAT are invaluable.

8.2 NP-complete problems

Hard problems, easy problems

In short, the world is full of search problems, some of which can be solved efficiently, while others seem to be very hard. This is depicted in the following table.

Hard problems (**NP**-complete)	Easy problems (in **P**)
3SAT	2SAT, HORN SAT
TRAVELING SALESMAN PROBLEM	MINIMUM SPANNING TREE
LONGEST PATH	SHORTEST PATH
3D MATCHING	BIPARTITE MATCHING
KNAPSACK	UNARY KNAPSACK
INDEPENDENT SET	INDEPENDENT SET on trees
INTEGER LINEAR PROGRAMMING	LINEAR PROGRAMMING
RUDRATA PATH	EULER PATH
BALANCED CUT	MINIMUM CUT

This table is worth contemplating. On the right we have problems that can be solved efficiently. On the left, we have a bunch of hard nuts that have escaped efficient solution over many decades or centuries.

The various problems on the right can be solved by algorithms that are specialized and diverse: dynamic programming, network flow, graph search, greedy. These problems are easy for a variety of different reasons.

In stark contrast, the problems on the left *are all difficult for the same reason!* At their core, they are all the same problem, just in different disguises! They are all *equivalent:* as we shall see in Section 8.3, each of them can be reduced to any of the others—and back.

P and NP

It's time to introduce some important concepts. We know what a search problem is: its defining characteristic is that any proposed solution can be quickly checked for correctness, in the sense that there is an efficient checking algorithm $\mathcal{C}$ that takes as input the given instance I (the data specifying the problem to be solved), as well as the proposed solution S, and outputs `true` if and only if S really is a solution to instance I. Moreover the running time of $\mathcal{C}(I, S)$ is bounded by a polynomial in $|I|$, the length of the instance. *We denote the class of all search problems by* **NP**.

We've seen many examples of **NP** search problems that are solvable in polynomial time. In such cases, there is an algorithm that takes as input an instance I and has a running time polynomial in $|I|$. If I has a solution, the algorithm returns such a solution; and if I has no solution, the algorithm correctly reports so. *The class of all search problems that can be solved in polynomial time is denoted* **P**. Hence, all the search problems on the right-hand side of the table are in **P**.

Why P and NP?

Okay, **P** must stand for "polynomial." But why use the initials **NP** (the common chatroom abbreviation for "no problem") to describe the class of search problems, some of which are terribly hard?

NP stands for "nondeterministic polynomial time," a term going back to the roots of complexity theory. Intuitively, it means that a solution to any search problem can be found and verified in polynomial time by a special (and quite unrealistic) sort of algorithm, called a *nondeterministic algorithm*. Such an algorithm has the power of *guessing* correctly at every step.

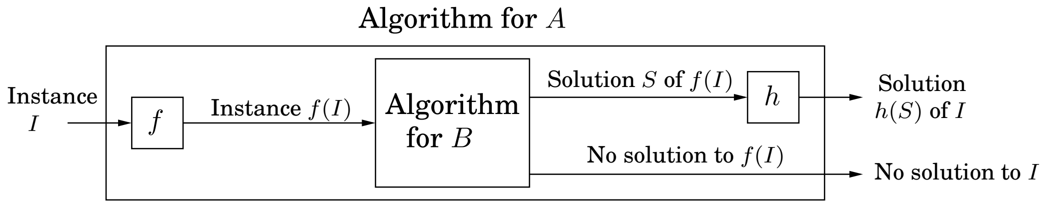
Incidentally, the original definition of **NP** (and its most common usage to this day) was not as a class of search problems, but as a class of *decision problems:* algorithmic questions that can be answered by yes or no. Example: "Is there a truth assignment that satisfies this Boolean formula?" But this too reflects a historical reality: At the time the theory of **NP**-completeness was being developed, researchers in the theory of computation were interested in formal languages, a domain in which such decision problems are of central importance.

Are there search problems that cannot be solved in polynomial time? In other words, is **P** $\neq$ **NP**? Most algorithms researchers think so. It is hard to believe that exponential search can always be avoided, that a simple trick will crack all these hard problems, famously unsolved for decades and centuries. And there is a good reason for mathematicians to believe that $\mathbf{P} \neq \mathbf{NP}$—the task of finding a proof for a given mathematical assertion is a search problem and is therefore in **NP** (after all, when a formal proof of a mathematical statement is written out in excruciating detail, it can be checked mechanically, line by line, by an efficient algorithm). So if $\mathbf{P} = \mathbf{NP}$, there would be an efficient method to prove any theorem, thus eliminating the need for mathematicians! All in all, there are a variety of reasons why it is widely believed that $\mathbf{P} \neq \mathbf{NP}$. However, proving this has turned out to be extremely difficult, one of the deepest and most important unsolved puzzles of mathematics.

Reductions, again

Even if we accept that $\mathbf{P} \neq \mathbf{NP}$, what about the specific problems on the left side of the table? On the basis of what evidence do we believe that these particular problems have no efficient algorithm (besides, of course, the historical fact that many clever mathematicians and computer scientists have tried hard and failed to find any)? Such evidence is provided by *reductions*, which translate one search problem into another. What they demonstrate is that the problems on the left side of the table are all, in some sense, *exactly the same problem*, except that they are stated in different languages. What's more, we will also use reductions to show that these problems are the *hardest* search problems in **NP**—if even one of them has a polynomial time algorithm, then *every* problem in **NP** has a polynomial time algorithm. Thus if we believe that $\mathbf{P} \neq \mathbf{NP}$, then all these search problems are hard.

We defined reductions in Chapter 7 and saw many examples of them. Let's now specialize this definition to search problems. A *reduction* from search problem A to search problem B is a polynomial-time algorithm f that transforms any instance I of A into an instance $f(I)$ of B, together with another polynomial-time algorithm h that maps any solution S of $f(I)$ back into a solution $h(S)$ of I; see the following diagram. If $f(I)$ has no solution, then neither does I. These two translation procedures f and h imply that any algorithm for B can be converted into an algorithm for A by bracketing it between f and h.

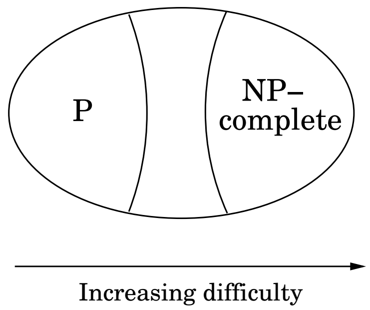


And now we can finally define the class of the hardest search problems.

A search problem is **NP**-*complete if all other search problems reduce to it.*

This is a very strong requirement indeed. For a problem to be **NP**-complete, it must be useful in solving every search problem in the world! It is remarkable that such problems exist. But they do, and the first column of the table we saw earlier is filled with the most famous examples. In Section 8.3 we shall see how all these problems reduce to one another, and also why all other search problems reduce to them.

Figure 8.6 The space **NP** of all search problems, assuming $\mathbf{P} \neq \mathbf{NP}$.



The two ways to use reductions

So far in this book the purpose of a reduction from a problem A to a problem B has been straightforward and honorable: We know how to solve B efficiently, and we want to use this knowledge to solve A. In this chapter, however, reductions from A to B serve a somewhat perverse goal: we know A is hard, and we use the reduction to prove that B is hard as well!

If we denote a reduction from A to B by

$$A \longrightarrow B$$

then we can say that *difficulty* flows in the direction of the arrow, while *efficient algorithms* move in the opposite direction. It is through this propagation of difficulty that we know **NP**-complete problems are hard: all other search problems reduce to them, and thus each **NP**-complete problem contains the complexity of all search problems. If even one **NP**-complete problem is in **P**, then $\mathbf{P} = \mathbf{NP}$.

Reductions also have the convenient property that they *compose*.

$$\text{If } A \longrightarrow B \text{ and } B \longrightarrow C\text{, then } A \longrightarrow C\ .$$

To see this, observe first of all that any reduction is completely specified by the pre- and postprocessing functions f and h (see the reduction diagram). If (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}) define the reductions from A to B and from B to C, respectively, then a reduction from A to C is given by compositions of these functions: $f_{BC} \circ f_{AB}$ maps an instance of A to an instance of C and $h_{AB} \circ h_{BC}$ sends a solution of C back to a solution of A.

This means that once we know a problem A is **NP**-complete, we can use it to prove that a new search problem B is also **NP**-complete, simply by reducing A to B. Such a reduction establishes that all problems in **NP** reduce to B, via A.

Factoring

One last point: we started off this book by introducing another famously hard search problem: FACTORING, the task of finding all prime factors of a given integer. But the difficulty of FACTORING is of a different nature than that of the other hard search problems we have just seen. For example, nobody believes that FACTORING is **NP**-complete. One major difference is that, in the case of FACTORING, the definition does not contain the now familiar clause "or report that none exists." A number can *always* be factored into primes.

Another difference (possibly not completely unrelated) is this: as we shall see in Chapter 10, FACTORING succumbs to the power of *quantum computation*—while SAT, TSP and the other **NP**-complete problems do not seem to.