

# A Crash Course on Discrete Probability

# Events and Probability

Consider a random process (e.g., throw a die, pick a card from a deck)

- Each possible outcome is a **simple event** (or sample point).
- The **sample space**  $\Omega$  is the set of all possible simple events.
- An **event** is a set of simple events (a subset of the sample space).
- With each simple event  $E$  we associate a real number  $0 \leq \Pr(E) \leq 1$  which is the **probability** of  $E$ .

# Probability Space

## Definition

A **probability space** has three components:

- 1 A **sample space**  $\Omega$ , which is the set of all possible outcomes of the random process modeled by the probability space;
- 2 A family of sets  $\mathcal{F}$  representing the allowable **events**, where each set in  $\mathcal{F}$  is a subset of the sample space  $\Omega$ ;
- 3 A **probability function**  $\Pr : \mathcal{F} \rightarrow \mathbf{R}$ , satisfying the definition below.

In a **discrete** probability space we use  $\mathcal{F} =$  “all the subsets of  $\Omega$ ”

# Probability Function

## Definition

A **probability function** is any function  $\Pr : \mathcal{F} \rightarrow \mathbf{R}$  that satisfies the following conditions:

- 1 For any event  $E$ ,  $0 \leq \Pr(E) \leq 1$ ;
- 2  $\Pr(\Omega) = 1$ ;
- 3 For any finite or countably infinite sequence of pairwise mutually disjoint events  $E_1, E_2, E_3, \dots$

$$\Pr \left( \bigcup_{i \geq 1} E_i \right) = \sum_{i \geq 1} \Pr(E_i).$$

The probability of an event is the sum of the probabilities of its simple events.

## Examples:

Consider the random process defined by the outcome of rolling a die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

We assume that all “facets” have equal probability, thus

$$\Pr(1) = \Pr(2) = \dots \Pr(6) = 1/6.$$

The probability of the event “odd outcome”

$$= \Pr(\{1, 3, 5\}) = 1/2$$

Assume that we roll two dice:

$\Omega =$  all ordered pairs  $\{(i, j), 1 \leq i, j \leq 6\}$ .

We assume that each (ordered) combination has probability  $1/36$ .

Probability of the event “sum = 2”

$$\Pr(\{(1, 1)\}) = 1/36.$$

Probability of the event “sum = 3”

$$\Pr(\{(1, 2), (2, 1)\}) = 2/36.$$

Let  $E_1 =$  “sum bounded by 6”,

$$E_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \\ (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$$

$$\Pr(E_1) = 15/36$$

Let  $E_2 =$  “both dice have odd numbers”,  $\Pr(E_2) = 1/4$ .

$$\Pr(E_1 \cap E_2) =$$

$$\Pr(\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (5, 1)\}) =$$

$$6/36 = 1/6.$$

# The union bound

## Theorem

Consider events  $E_1, E_2, \dots, E_n$ . Then we have

$$\Pr\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \Pr(E_i).$$

Example: I roll a die:

- Let  $E_1 =$  “result is odd”
- Let  $E_2 =$  “result is  $\leq 2$ ”



## The Monty Hall problem



# The Monty Hall problem - Sample space

Let's assume that

- The car is in a random door.
- The player chooses a random door.
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We can define as a sample space the set of triples  $(p, c, s)$ :

- $p$ : The door with the car
- $c$ : The door the player chooses
- $s$ : The door opened by the presenter

$$\Omega = \{(1, 1, 2), (1, 1, 3), (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 2, 1), (2, 2, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (3, 3, 1), (3, 3, 2)\}$$

# The Monty Hall problem - Probabilities

Event	Prob.	Stay Wins	Switch Wins
(1, 1, 2)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(1, 1, 3)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(1, 2, 3)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(1, 3, 2)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(2, 1, 3)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(2, 2, 1)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(2, 2, 3)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(2, 3, 1)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 1, 2)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 2, 1)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 3, 1)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(3, 3, 2)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
		$6/18 = 1/3$	$6/9 = 2/3$

# Independent Events

## Definition

Two events  $E$  and  $F$  are **independent** if and only if

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F).$$

## Independent Events, examples

Example: You pick a card from a deck.

- $E =$  "Pick an ace"
- $F =$  "Pick a heart"

Example: You roll a die

- $E =$  "number is even"
- $F =$  "number is  $\leq 4$ "

Basically, two events are independent if when one happens it doesn't tell you anything about if the other happened.

# Conditional Probability

What is the probability that a random student at Sapienza was born in Roma.

$E_1$  = the event “born in Roma.”

$E_2$  = the event “a student in Sapienza.”

The conditional probability that a a student at Sapienza was born in Roma is written:

$$\Pr(E_1 | E_2).$$

# Computing Conditional Probabilities

## Definition

The **conditional probability** that event  $E$  occurs given that event  $F$  occurs is

$$\Pr(E | F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

The conditional probability is only well-defined if  $\Pr(F) > 0$ .

By conditioning on  $F$  we restrict the sample space to the set  $F$ . Thus we are interested in  $\Pr(E \cap F)$  “normalized” by  $\Pr(F)$ .



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$$\Pr(E_2) = 1/2 = 18/36.$$

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)} = \frac{5/36}{1/2} = 5/18.$$

## Question

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Define the events:

$E =$  "One of the two children is a boy"  $= \{BB, BG, GB\}$ ,

$F =$  "Both children are boys"  $= \{BB\}$

$$\Pr(F | E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{\Pr(\{BB\})}{\Pr(\{BB, BG, GB\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

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**Answer:**

- ①  $1/3$
- ②  $1/2$
- ③ Something close to  $1/3$ , but not exactly
- ④ Something close to  $1/2$ , but not exactly

## Example - a posteriori probability

We are given 2 coins:

- one is a fair coin  $A$
- the other coin,  $B$ , has head on both sides

We choose a coin at random, i.e. each coin is chosen with probability  $1/2$ . We then flip the coin.

Given that we got head, what is the probability that we chose the fair coin  $A$ ???

Define a sample space of ordered pairs (*coin*, *outcome*).  
The sample space has three points

$$\{(A, h), (A, t), (B, h)\}$$

$$\Pr((A, h)) = \Pr((A, t)) = 1/4$$

$$\Pr((B, h)) = 1/2$$

Define two events:

$E_1$  = "Chose coin *A*".

$E_2$  = "Outcome is head".

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)} = \frac{1/4}{1/4 + 1/2} = 1/3.$$

# Independence

Two events  $A$  and  $B$  are independent if

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B),$$

or

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A).$$

## A Useful Identity

What is the probability that *Juventus* will win *Ajax* in the game they play tomorrow?

Assume we know that (numbers made up!)

- the probability that *Juventus* will win *Ajax* is 62% if tomorrow it does not rain.
- the probability that *Juventus* will win *Ajax* is 40% if tomorrow it rains.
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Assume two events  $A$  and  $B$ .

$$\begin{aligned}\Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B^c) \\ &= \Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid B^c) \cdot \Pr(B^c)\end{aligned}$$



# Random Variable

## Definition

A **random variable**  $X$  on a sample space  $\Omega$  is a function on  $\Omega$ ; that is,  $X : \Omega \rightarrow \mathcal{R}$ .

A **discrete random variable** is a random variable that takes on only a finite or countably infinite number of values.

## Examples:

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- ⑤ I pick 10 random students,  $X = \text{"average weight"}$
- ⑥  $X = \text{"Running time of quicksort"}$

# Independent random variables

## Definition

Two random variables  $X$  and  $Y$  are **independent** if and only if

$$\Pr((X = x) \cap (Y = y)) = \Pr(X = x) \cdot \Pr(Y = y)$$

for all values  $x$  and  $y$ .



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- I pick a random card from a deck. The **value** that I got and the **suit** that I got are independent.
- I pick a random person in Rome. The **age** and the **weight** are **not** independent.

# Expectation

## Definition

The **expectation** of a discrete random variable  $X$ , denoted by  $\mathbf{E}[X]$ , is given by

$$\mathbf{E}[X] = \sum_i i \Pr(X = i),$$

where the summation is over all values in the range of  $X$ .

## Examples:

- The expected value of one die roll is:

$$E[X] = \sum_{i=1}^6 i \Pr(X = i) = \sum_{i=1}^6 \frac{i}{6} = 3.5.$$

- The expectation of the random variable  $X$  representing the sum of two dice is

$$E[X] = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \dots + \frac{1}{36} \cdot 12 = 7.$$

- Let  $X$  take on the value  $2^i$  with probability  $1/2^i$  for  $i = 1, 2, \dots$

$$E[X] = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \sum_{i=1}^{\infty} 1 = \infty.$$

Consider a game in which a player chooses a number in  $\{1, 2, \dots, 6\}$  and then rolls 3 dice.

The player wins \$1 for each die that matches the number, he loses \$1 if no die matches the number.

What is the expected outcome of that game:

Consider a game in which a player chooses a number in  $\{1, 2, \dots, 6\}$  and then rolls 3 dice.

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What is the expected outcome of that game:

$$-1\left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)^3 = -\frac{17}{216}.$$

# Linearity of Expectation

## Theorem

For any two random variables  $X$  and  $Y$

$$E[X + Y] = E[X] + E[Y].$$

## Theorem

For any constant  $c$  and discrete random variable  $X$ ,

$$E[cX] = cE[X].$$

Note:  $X$  and  $Y$  do not have to be independent.



## Examples:

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- The expectation of the outcome of one die plus twice the outcome of a second die is. . .

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- Instead we define  $N$  0-1 random variables  $X_i$ :

$$X_i = \begin{cases} 1, & \text{if person } i \text{ got his coat,} \\ 0, & \text{otherwise} \end{cases}$$

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- $E[X_i] = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) =$
- $\Pr(X_i = 1) = \frac{1}{N}$
- $E[X] = E \left[ \sum_{i=1}^N X_i \right] = \sum_{i=1}^N E[X_i] = 1$

# Bernoulli Random Variable

A **Bernoulli** or an **indicator** random variable:

$$Y = \begin{cases} 1 & \text{if the experiment succeeds,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[Y] = p \cdot 1 + (1 - p) \cdot 0 = p = \mathbf{Pr}(Y = 1).$$



# Binomial Random Variable

Assume that we repeat  $n$  independent Bernoulli trials that have probability  $p$ .

Examples:

- I flip  $n$  coins,  $X_i = 1$ , if the  $i$ th flip is “head” ( $p = 1/2$ )
- I roll  $n$  dice,  $X_i = 1$ , if the  $i$ th die roll is a 4 ( $p = 1/6$ )
- I choose  $n$  cards,  $X_i = 1$ , if the  $i$ th card is a  $J, Q, K$  ( $p = 12/52$ .)

Let  $X = \sum_{i=1}^n X_i$ .

$X$  is a Binomial random variable.

# Binomial Random Variable

## Definition

A binomial random variable  $X$  with parameters  $n$  and  $p$ , denoted by  $B(n, p)$ , is defined by the following probability distribution on  $j = 0, 1, 2, \dots, n$ :

$$\Pr(X = j) = \binom{n}{j} p^j (1 - p)^{n-j}.$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$  is the number of ways that we can select  $k$  elements out of  $n$ .

## Expectation of a Binomial Random Variable

$$\begin{aligned}\mathbf{E}[X] &= \sum_{j=0}^n j \Pr(X = j) \\ &= \sum_{j=0}^n j \binom{n}{j} p^j (1-p)^{n-j}\end{aligned}$$

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## Expectation of a Binomial R. V. - 2nd way

Using linearity of expectations

$$\mathbf{E}[X] = \mathbf{E} \left[ \sum_{i=1}^n X_i \right] = \sum_{i=1}^n \mathbf{E}[X_i] = np.$$