# Course : Data mining

## Lecture : Mining data streams

Aristides Gionis
Department of Computer Science
Aalto University

visiting in Sapienza University of Rome
fall 2016

# reading assignment

- LRU book: chapter 4
- optional reading
  - paper by Alon, Matias, and Szegedy [Alon et al., 1999]
  - paper by Charikar, Chen, and Farach-Colton [Charikar et al., 2002]
  - paper by Cormode and Muthukrishnan [Cormode and Muthukrishnan, 2005]

# data streams

- a data stream is a massive sequence of data
- too large to store (on disk, memory, cache, etc.)
- examples:
  - social media (e.g., twitter feed, foursquare checkins)
  - sensor networks (weather, radars, cameras, etc.)
  - network traffic (trajectories, source/destination pairs)
  - satellite data feed
- how to deal with such data?
- what are the issues?

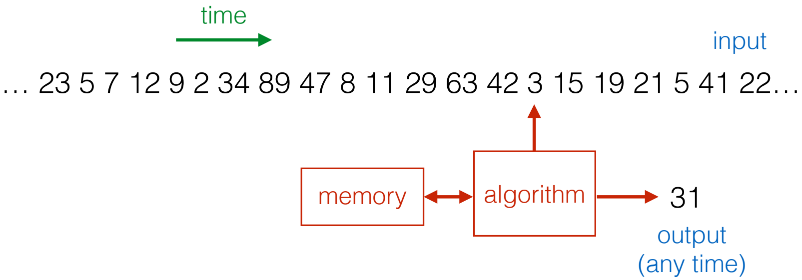
# issues when working with data streams

- space
  - data size is very large
  - often not possible to store the whole dataset
  - inspect each data item, make some computations, do not store it, and never get to inspect it again
  - sometimes data is stored, but making one single pass takes a lot of time, especially when the data is stored on disk
  - can afford a small number of passes over the data
- time
  - data "flies by" at a high speed
  - computation time per data item needs to be small

# data streams

- data items can be of complex types
  - documents (tweets, news articles)
  - images
  - geo-located time-series
  - ...
- to study basic algorithmic ideas we abstract away application-specific details
- consider the data stream as a sequence of numbers

# data-stream model



# data-stream model

- stream: $m$ elements from universe of size $n$, e.g.,

$$\langle x_1, x_2, \ldots, x_m \rangle = 6, 1, 7, 4, 9, 1, 5, 1, 5, \ldots$$

- goal: compute a function over the elements of the stream, e.g., median, number of distinct elements, quantiles, ...
- constraints:
  1. limited working memory, sublinear in $n$ and $m$ e.g., $\mathcal{O}(\log n + \log m)$,
  2. access data sequentially
  3. limited number of passes, in some cases only one
  4. process each element quickly, e.g., $\mathcal{O}(1)$, $\mathcal{O}(\log n)$, etc.

# warm up: computing some simple functions

- assume that a number can be stored in $\mathcal{O}(\log n)$ space
- max, min can be computed with $\mathcal{O}(\log n)$ space
- sum, mean (average) need $\mathcal{O}(\log n + \log m)$ space

$$\mu_X = \mathbb{E}[X] = \mathbb{E}[x_1, \ldots, x_m] = \frac{1}{m}\sum_{i=1}^{m} x_i$$

- what about variance?

$$\begin{aligned}\mathbb{V}ar[X] = \mathbb{V}ar[x_1, \ldots, x_m] &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \frac{1}{m}\sum_{i=1}^{m}(x_i - \mu_X)^2\end{aligned}$$

- two passes? one pass?

# how to tackle massive data streams?

- a general and powerful technique: sampling
- idea:
  1. keep a random sample of the data stream
  2. perform the computation on the sample
  3. extrapolate
- example: compute the median of a data stream

  (how to extrapolate in this case?)

- but ... how to keep a random sample of a data stream?

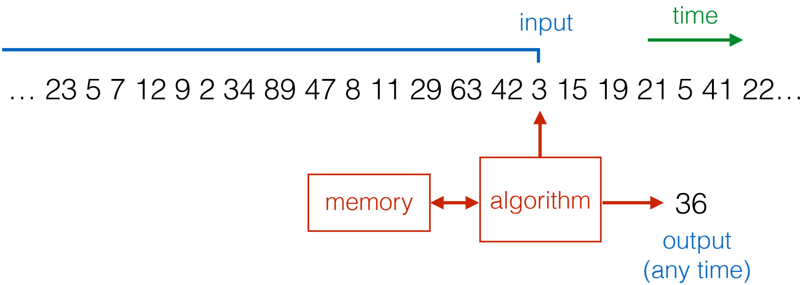
# reservoir sampling

- problem: take a uniform sample $s$ from a stream of unknown length
- algorithm:
  - initially $s \leftarrow x_1$
  - on seeing the $t$-th element, $s \leftarrow x_t$ with probability $1/t$
- analysis:
  - what is the probability that $s = x_i$ at some time $t \geq i$?

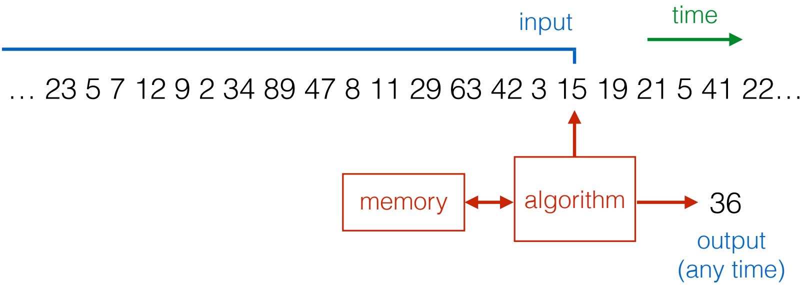
$$\begin{aligned}\Pr[s = x_i] &= \frac{1}{i} \cdot \left(1 - \frac{1}{i+1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{t-1}\right) \cdot \left(1 - \frac{1}{t}\right) \\ &= \frac{1}{i} \cdot \frac{i}{i+1} \cdot \ldots \cdot \frac{t-2}{t-1} \cdot \frac{t-1}{t} = \frac{1}{t}\end{aligned}$$

- how much space? $\mathcal{O}(\log n)$
- to get $k$ samples we need $\mathcal{O}(k \log n)$ bits

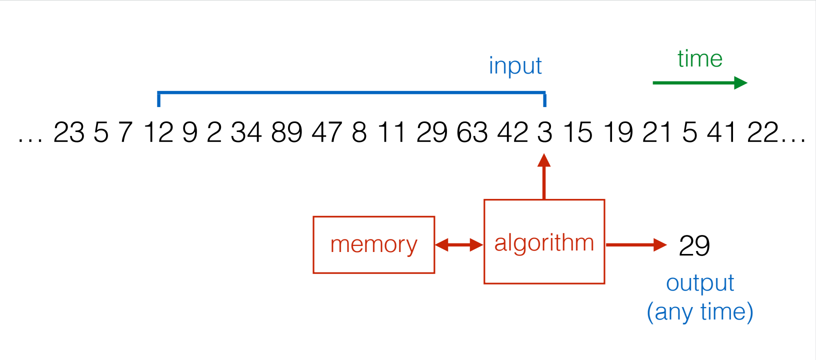
# infinite data-stream model



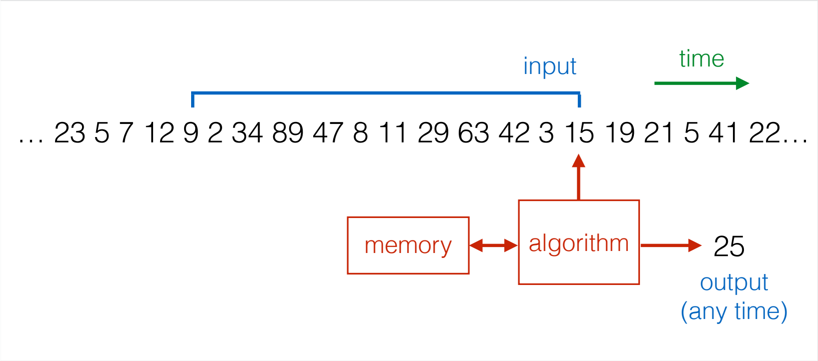
# infinite data-stream model



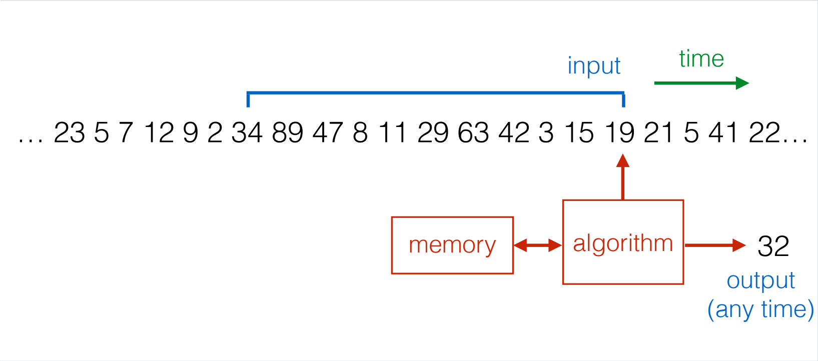
# sliding-window data-stream model



# sliding-window data-stream model



# sliding-window data-stream model



# sliding-window data-stream model

- does sliding-window model makes computation easier or harder?
- how to compute `sum`?
- how to keep a random sample?

- all computations can be done with $\mathcal{O}(w)$ space
- can we do better?

# priority sampling for sliding window

- maintain a uniform sample from the last $w$ items
- reservoir sampling does not work in this model
- algorithm:
  1. for each $x_i$ we pick a random value $v_i \in (0, 1)$
  2. for window $\langle x_{j-w+1}, \ldots, x_j \rangle$ return $x_i$ with smallest $v_i$

  - to do this, maintain set of all elements in sliding window whose $v$ value is minimal among all subsequent values

# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

.64 .12 .31 .84 .27 .56 .91

# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

.64 .12 .31 .84 .27 .56 .91

# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

.64 .12 .31 .84 .27 .56 .91

# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

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# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

.64 .12 .31 .84 .27 .56 .91 .42

# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

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# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

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# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

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# priority sampling for sliding window

… 23 5 7 12 9 2 34 89 47 8 11 29 63 …

.64 .12 .31 .84 .27 .56 .91 .42 .73 .20

# priority sampling for sliding window

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| … 23 | 5 | 7 | 12 | 9 | 2 | 34 | 89 | 47 | 8 | 11 | 29 | 63 … |
| .64 | .12 | .31 | .84 | .27 | .56 | .91 | .42 | .73 | .20 | | | |

# priority sampling for sliding window

- correctness 1: in any given window each item has equal chance to be selected as a random sample
- correctness 2: each removed minimal element has a smaller element that comes after

- space efficiency: how many minimal elements do we expect at any given point?
- $\mathcal{O}(\log w)$
- so, expected space requirement is $\mathcal{O}(\log w \log n)$

- time efficiency: maintaining list of minimal elements requires $\mathcal{O}(\log w)$ time

# mining data streams

- what are real-world applications?
- imagine monitoring a social feed stream
  - a stream of hashtags in twitter
  - what are interesting questions to ask?
  - do data stream considerations (space/time) really matter?

# how to tackle massive data streams?

- a general and powerful technique: sketching
- general idea:
- apply a linear projection that takes high-dimensional data to a smaller dimensional space
- post-process lower dimensional image to estimate the quantities of interest

# computing statistics on data streams

- $X = (x_1, x_2, \ldots, x_m)$ a sequence of elements
- each $x_i$ is a member of the set $N = \{1, \ldots, n\}$
- $m_i = |\{j : x_j = i\}|$ the number of occurrences of $i$
- define the $k$-th frequency moment

$$F_k = \sum_{i=1}^{n} m_i^k$$

- $F_0$ is the number of distinct elements
- $F_1$ is the length of the sequence
- $F_2$ is the second moment: index of homogeneity, size of self-join, and other applications
- $F_\infty^*$ frequency of most frequent element

# computing statistics on data streams

- how much space I need to compute the frequency moments in a straighforward manner?
- how to compute the frequency moments using less than $O(n \log m)$ space?
- problem studied by Alon, Matias, Szegedy [Alon et al., 1999]
- sketching: create a sketch that takes much less space and gives an estimation of $F_k$

# estimating the number of distinct values ($F_0$)

[Flajolet and Martin, 1985]

- consider a bit vector of length $O(\log n)$
- initialize all bits to 0
- upon seen $x_i$, set:
  - the 1-st bit with probability $1/2$
  - the 2-nd bit with probability $1/4$
  - ...
  - the $i$-th bit with probability $1/2^i$
- important: bits are set deterministically for each $x_i$
- let $R$ be the index of the largest bit set
- return $Y = 2^R$

# estimating the number of distinct values ($F_0$)

[Flajolet and Martin, 1985]

intuition:

- the $i$-th bit is set with probability $1/2^i$
- e.g., after seeing roughly 32 distinct elements, we expect to get the 5-th bit set
- if the bit vector is 0000011111 the estimate is 32

# estimating number of distinct values ($F_0$)

Theorem. For every $c > 2$, the algorithm computes a number $Y$ using $\mathcal{O}(\log n)$ memory bits, such that the probability that the ratio between $Y$ and $F_0$ is not between $1/c$ and $c$ is at most $2/c$.

# estimating $F_2$

- $X = (x_1, x_2, \ldots, x_m)$ a sequence of elements
- each $x_i$ is a member of the set $N = \{1, \ldots, n\}$
- $m_i = |\{j : x_j = i\}|$ the number of occurrences of $i$
- $F_k = \sum_{i=1}^{n} m_i^k$

- algorithm:
- hash each $i \in \{1, \ldots, n\}$ to a random $\epsilon_i \in \{-1, +1\}$
- maintain sketch $Z = \sum_i \epsilon_i m_i$
  just need space $\mathcal{O}(\log n + \log m)$
- take $X = Z^2$
- return the average $Y$ of $k$ such estimates $X_1, \ldots, X_k$
- $Y = \frac{1}{k} \sum_{j=1}^{k} X_j$ where $k = \frac{16}{\lambda^2}$

# expectation of the estimate is correct

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= \mathbb{E}\left[Z^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n} \epsilon_i m_i\right)^2\right] \\
&= \sum_{i=1}^{n} m_i^2 \mathbb{E}\left[\epsilon_i^2\right] + 2 \sum_{i<j} m_i m_j \mathbb{E}\left[\epsilon_i\right] \mathbb{E}\left[\epsilon_j\right] \\
&= \sum_{i=1}^{n} m_i^2 = F_2
\end{aligned}
$$

# accuracy of the estimate

easy to show

$$\mathbb{E}\left[X^2\right] = \sum_{i=1}^{n} m_i^4 + 6 \sum_{i<j} m_i^2 m_j^2$$

which gives

$$\mathbb{V}ar\left[X\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 = 4 \sum_{i<j} m_i^2 m_j^2 \leq 2F_2^2$$

and by Chebyshev's inequality

$$\Pr[|Y - F_2| \geq \lambda F_2] \leq \frac{\mathbb{V}ar\left[Y\right]}{\lambda^2 F_2^2} = \frac{\mathbb{V}ar\left[X\right]/k}{\lambda^2 F_2^2} \leq \frac{2F_2^2/k}{\lambda^2 F_2^2} = \frac{2}{k\lambda^2} = \frac{1}{8}$$

# finding frequent items in a data stream

- optional reading :
  paper by Charikar, Chen, and Farach-Colton
  [Charikar et al., 2002]

# finding frequent items in a data stream

- consider again a data stream
- $X = (x_1, x_2, \ldots, x_m)$ a data stream
- each $x_i$ is a member of the set $N = \{1, \ldots, n\}$
- $m_i = |\{j : x_j = i\}|$ the number of occurrences of $i$
- $f_i = m_i/m$ the frequency of item $i$

- problem : estimate most frequent items in data stream

# finding frequent items in a data stream

- problem formalization
- rename items $\{o_1, \ldots, o_n\}$ so that $m_1 \geq \ldots \geq m_n$
- given $k < n$ want to return top-$k$ items $o_1, \ldots, o_k$

# finding frequent items in a data stream

- problem formalization — first attempt

- problem $\textsc{FindCandidateTop}(X, k, \ell)$
  - given stream $X$ and integers $k$ and $\ell$
  - return list of $\ell$ items, so that top most frequent $k$ items of $X$ occur in the list

- should return all most frequent items

# finding frequent items in a data stream

- $\textsc{FindCandidateTop}(X, k, \ell)$ can be too hard to solve
- consider the case $m_k = m_{\ell+1} + 1$
  - i.e., number of occurences of $k$-th frequent item exceeds only by 1 the number of occurences of the $(\ell + 1)$-th frequent item
- almost impossible to find a list that contains the $k$ most frequent items

# finding frequent items in a data stream

- problem formalization — second attempt

- problem $\textsc{FindApproxTop}(X, k, \epsilon)$
  - given stream $X$, integer $k$, and real $\epsilon < 1$
  - return list of $k$ items, so that for each item $i$ in the list it is $m_i \geq (1-\epsilon) m_k$

- no guarantee to return all most frequent items, but if return an item it should be frequent enough

# finding frequent items in a data stream

- problem : $\textsc{FindCandidateTop}(X, k, \ell)$
- algorithm : $\textsc{Sampling}$
- modification of reservoir sampling
  - keep a list of sampled items, plus a counter for each item
  - if an item is sampled again, increment its counter

# analysis of SAMPLING algorithm

- let $x$ the number of items need to keep in the sample
- probability to be included in the sample is $x/m$
- want to ensure that $o_k$ appears in the sample
- need to set $x/m$ at least $\mathcal{O}((\log m)/m_k)$
- so $x$ should be at least $\mathcal{O}((\log m)/f_k)$
- so we have solution for $\textsc{FindCandidateTop}(X, k, \mathcal{O}((\log m)/f_k))$
- limitation : it requires knowing $m$ and $f_k$

# finding frequent items in a data stream

- problem : $\textsc{FindApproxTop}(X, k, \epsilon)$

- algorithm : $\textsc{CountSketch}$
  - based on sketching techniques

- intuition
  - use a hash function $s$ and a counter $c$
  - function $s$ hashes objects to $\{-1, +1\}$
  - for each item $o_i$ seen in the stream, set $c \leftarrow c + s[o_i]$
  - then $\mathbb{E}\left[c \cdot s[o_i]\right] = m_i$ (prove it!)
  - so, estimate $m_i$ by $c \cdot s[o_i]$

# the CountSketch algorithm

- problem with using one hash function and one counter
  - very high variance
- remedy 1

  use $t$ hash functions $s_1, \ldots, s_t$ and $t$ counters $c_1, \ldots, c_t$
  - for each item $o_i$ seen in the stream,

    set $c_j \leftarrow c_j + s_j[o_i]$, for all $j = 1, \ldots, t$
  - to estimate $m_i$ take median of $\{c_1 \cdot s_1[o_i], \ldots, c_t \cdot s_t[o_i]\}$

    (as before $\mathbb{E}\left[c_j \cdot s_j[o_i]\right] = m_i$ for all $j = 1, \ldots, t$)

# the CountSketch algorithm

- problem with previous idea
  - high-frequency items (e.g., $o_1$) may spoil estimates of lower-frequency items (e.g., $o_k$)
- remedy 2
  - do not update all counters with all items
  - replace each counter with a hash table of $b$ counters
  - items update different subsets of counters, one per hash table
  - each item gets enough high-confidence estimates (those avoiding collisions with high-frequency elements)

# the CountSketch algorithm

- use parameters $t$ and $b$
- let $h_1, \ldots, h_t$ be hash functions from items to $1, \ldots, b$
- let $s_1, \ldots, s_t$ be hash functions from items to $\{-1, +1\}$
- consider $t \times b$ table of counters
- for each item $o_i$ seen in the stream,
  set $h_j[o_i] \leftarrow h_j[o_i] + s_j[o_i]$, for all $j = 1, \ldots, t$
- to estimate $m_i$ take median of
  $\{h_1[o_i] \cdot s_1[o_i], \ldots, h_t[o_i] \cdot s_t[o_i]\}$

# an improved data stream summary

- the CountMinSketch data stream summary
- optional reading
  paper by Cormode and Muthukrishnan
  [Cormode and Muthukrishnan, 2005]

# the CountMinSketch data stream summary

- limitations of existing sketches
- – model limitations (a sequence of items / numbers)
- – space required is $\mathcal{O}(\frac{1}{\epsilon^2})$

  recall that quarantees are quantified by $\epsilon$, $\delta$ parameters

  $\epsilon$ : accuracy

  $\delta$ : probability of failure
- – update time proportional to the whole sketch
- – different sketch for each summary

- CountMinSketch addresses all those limitations

# incremental data-stream model

- consider a vector $\mathbf{x}(t) = \{x_1(t), \ldots, x_n(t)\}$
- number of coordinates $n$ potentially very large
- $\mathbf{x}(t)$ the values of vector at time $t$
- at each time $t$ a vector coordinate is updated
- data stream : updates $(i_t, c_t)$ for $t = 1, \ldots$
- then

$$x_{i_t}(t) \leftarrow x_{i_t}(t-1) + c_t$$

and

$$x_j(t) \leftarrow x_j(t-1), \text{ for } j \neq i_t$$

# incremental data-stream model

- generalization of previous model

  previous model was $c_t = 1$

- special cases

– cash register model : $c_t \geq 0$

– turnstile model : $c_t$ can be negative

  – non-negative turnstile model : $x_i(t) \geq 0$

  – general turnstile model : $x_i(t)$ can be negative

# the CountMinSketch data stream summary

- interesting queries that we would like to handle

– point query $\mathcal{Q}(i)$ : approximate $x_i$

– range query $\mathcal{Q}(\ell, r)$ : approximate $\sum_{i=\ell}^{r} x_i$

– inner product $\mathcal{Q}(\mathbf{x}, \mathbf{y})$ : approximate $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i\, y_i$

– $\phi$-quantiles

– heavy-hitters : most frequent items

given frequency threshold $\phi$, find items $i$ for which

$x_i \geq (\phi - \epsilon)||\mathbf{x}||_1$ for some $\epsilon < \phi$

# the CountMinSketch data structure

- similar to CountSketch
- a table of counters $C$ of dimension $d \times w$
- $d$ hash functions $h_1, .., h_d$ from $\{1, .., n\}$ to $\{1, .., w\}$ chosen from a pairwise-independent family

$$C = \begin{pmatrix} C[1,1] & \cdots & C[1,w] \\ \vdots & \ddots & \vdots \\ C[d,1] & \cdots & C[d,w] \end{pmatrix}$$

- parameters $d$ and $w$ specify the space requirements depend on error bounds we want to achieve

# CountMinSketch : update summary

- given $(i_t, c_t)$ update one counter in each row of $C$, in particular

$$C[j, h_j(i_t)] \leftarrow C[j, h_j(i_t)] + c_t$$

for all $j = 1, \ldots, d$

# CountMinSketch : point query

- the answer to $\mathcal{Q}(i)$ is $\hat{x}_i = \min_j C[j, h_j(i)]$

- theorem : the estimate $\hat{x}_i$ satisfies

  (i) $x_i \leq \hat{x}_i$

  (ii) $\hat{x}_i \leq x_i + \epsilon||\mathbf{x}||_1$ with prob at least $1 - \delta$

# CountMinSketch

- similar type of estimates for other queries
  - range, inner product, etc.

- parameters are set to

$$d = \left\lceil \log \frac{1}{\delta} \right\rceil \quad \text{and} \quad w = \left\lceil \frac{1}{\epsilon} \right\rceil$$

  - improved space ; instead of usual $\mathcal{O}(\frac{1}{\epsilon^2})$
  - improved update time : access only $d$ counters

# references I


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