

A Crash Course on Discrete Probability

Events and Probability

Consider a random process (e.g., throw a die, pick a card from a deck)

- Each possible outcome is a **simple event** (or sample point).
- The **sample space** Ω is the set of all possible simple events.
- An **event** is a set of simple events (a subset of the sample space).
- With each simple event E we associate a real number $0 \leq \Pr(E) \leq 1$ which is the **probability** of E .

Probability Space

Definition

A **probability space** has three components:

- 1 A **sample space** Ω , which is the set of all possible outcomes of the random process modeled by the probability space;
- 2 A family of sets \mathcal{F} representing the allowable **events**, where each set in \mathcal{F} is a subset of the sample space Ω ;
- 3 A **probability function** $\Pr : \mathcal{F} \rightarrow \mathbf{R}$, satisfying the definition below.

In a **discrete** probability space we use $\mathcal{F} =$ “all the subsets of Ω ”

Probability Function

Definition

A **probability function** is any function $\Pr : \mathcal{F} \rightarrow \mathbf{R}$ that satisfies the following conditions:

- 1 For any event E , $0 \leq \Pr(E) \leq 1$;
- 2 $\Pr(\Omega) = 1$;
- 3 For any finite or countably infinite sequence of pairwise mutually disjoint events E_1, E_2, E_3, \dots

$$\Pr \left(\bigcup_{i \geq 1} E_i \right) = \sum_{i \geq 1} \Pr(E_i).$$

The probability of an event is the sum of the probabilities of its simple events.

Examples:

Consider the random process defined by the outcome of rolling a die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

We assume that all “facets” have equal probability, thus

$$\Pr(1) = \Pr(2) = \dots \Pr(6) = 1/6.$$

The probability of the event “odd outcome”

$$= \Pr(\{1, 3, 5\}) = 1/2$$

Assume that we roll two dice:

$\Omega =$ all ordered pairs $\{(i, j), 1 \leq i, j \leq 6\}$.

We assume that each (ordered) combination has probability $1/36$.

Probability of the event “sum = 2”

$$\Pr(\{(1, 1)\}) = 1/36.$$

Probability of the event “sum = 3”

$$\Pr(\{(1, 2), (2, 1)\}) = 2/36.$$

Let $E_1 =$ “sum bounded by 6”,

$$E_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \\ (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$$

$$\Pr(E_1) = 15/36$$

Let $E_2 =$ “both dice have odd numbers”, $\Pr(E_2) = 1/4$.

$$\Pr(E_1 \cap E_2) =$$

$$\Pr(\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (5, 1)\}) =$$

$$6/36 = 1/6.$$

The union bound

Theorem

Consider events E_1, E_2, \dots, E_n . Then we have

$$\Pr\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \Pr(E_i).$$

Example: I roll a die:

- Let $E_1 =$ “result is odd”
- Let $E_2 =$ “result is ≤ 2 ”

The Monty Hall problem



The Monty Hall problem - Sample space

Let's assume that

- The car is in a random door.
- The player chooses a random door.
- Whenever there's a choice the presenter chooses a random door.

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We can define as a sample space the set of triples (p, c, s) :

- p : The door with the car
- c : The door the player chooses
- s : The door opened by the presenter

$$\Omega = \{(1, 1, 2), (1, 1, 3), (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 2, 1), (2, 2, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (3, 3, 1), (3, 3, 2)\}$$

The Monty Hall problem - Probabilities

Event	Prob.	Stay Wins	Switch Wins
(1, 1, 2)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(1, 1, 3)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(1, 2, 3)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(1, 3, 2)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(2, 1, 3)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(2, 2, 1)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(2, 2, 3)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(2, 3, 1)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 1, 2)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 2, 1)	$\frac{1}{3} \cdot \frac{1}{3}$		✓
(3, 3, 1)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
(3, 3, 2)	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$	✓	
		$6/18 = 1/3$	$6/9 = 2/3$

Independent Events

Definition

Two events E and F are **independent** if and only if

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F).$$

Independent Events, examples

Example: You pick a card from a deck.

- $E =$ "Pick an ace"
- $F =$ "Pick a heart"

Example: You roll a die

- $E =$ "number is even"
- $F =$ "number is ≤ 4 "

Basically, two events are independent if when one happens it doesn't tell you anything about if the other happened.

Conditional Probability

What is the probability that a random student at La Sapienza was born in Roma.

E_1 = the event “born in Roma.”

E_2 = the event “a student in La Sapienza.”

The conditional probability that a a student at Sapienza was born in Roma is written:

$$\Pr(E_1 | E_2).$$

Computing Conditional Probabilities

Definition

The **conditional probability** that event E occurs given that event F occurs is

$$\Pr(E | F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

The conditional probability is only well-defined if $\Pr(F) > 0$.

By conditioning on F we restrict the sample space to the set F . Thus we are interested in $\Pr(E \cap F)$ “normalized” by $\Pr(F)$.

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$$\Pr(E_2) = 1/2 = 18/36.$$

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)} = \frac{5/36}{1/2} = 5/18.$$

Example - a posteriori probability

We are given 2 coins:

- one is a fair coin A
- the other coin, B , has head on both sides

We choose a coin at random, i.e. each coin is chosen with probability $1/2$. We then flip the coin.

Given that we got head, what is the probability that we chose the fair coin A ???

Define a sample space of ordered pairs (*coin*, *outcome*).
The sample space has three points

$$\{(A, h), (A, t), (B, h)\}$$

$$\Pr((A, h)) = \Pr((A, t)) = 1/4$$

$$\Pr((B, h)) = 1/2$$

Define two events:

E_1 = "Chose coin *A*".

E_2 = "Outcome is head".

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)} = \frac{1/4}{1/4 + 1/2} = 1/3.$$

Independence

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B),$$

or

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A).$$

A Useful Identity

Assume two events A and B .

$$\begin{aligned}\Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B^c) \\ &= \Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid B^c) \cdot \Pr(B^c)\end{aligned}$$

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Define the event B that “the random person is a man.”

Random Variable

Definition

A **random variable** X on a sample space Ω is a function on Ω ; that is, $X : \Omega \rightarrow \mathcal{R}$.

A **discrete random variable** is a random variable that takes on only a finite or countably infinite number of values.

Examples:

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In practice, a random variable is some random quantity that we are interested in:

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- 3 Consider a gambling game in which a player flips two coins, if he gets heads in both coins he wins \$3, else he loses \$1. The payoff of the game is a random variable.
- 4 I pick a card, $X = \begin{cases} 1, & \text{if card is an Ace} \\ 0, & \text{otherwise} \end{cases}$
- 5 I pick 10 random students, $X = \text{"average weight"}$
- 6 $X = \text{"Running time of quicksort"}$

Independent random variables

Definition

Two random variables X and Y are **independent** if and only if

$$\Pr((X = x) \cap (Y = y)) = \Pr(X = x) \cdot \Pr(Y = y)$$

for all values x and y .

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- I pick a random card from a deck. The **value** that I got and the **suit** that I got are independent.
- I pick a random person in Rome. The **age** and the **weight** are **not** independent.

Expectation

Definition

The **expectation** of a discrete random variable X , denoted by $\mathbf{E}[X]$, is given by

$$\mathbf{E}[X] = \sum_i i \Pr(X = i),$$

where the summation is over all values in the range of X .

Examples:

- The expected value of one die roll is:

$$E[X] = \sum_{i=1}^6 i \Pr(X = i) = \sum_{i=1}^6 i \frac{1}{6} = 3\frac{1}{2}.$$

- The expectation of the random variable X representing the sum of two dice is

$$E[X] = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \dots + \frac{1}{36} \cdot 12 = 7.$$

- Let X take on the value 2^i with probability $1/2^i$ for $i = 1, 2, \dots$

$$E[X] = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \sum_{i=1}^{\infty} 1 = \infty.$$

Consider a game in which a player chooses a number in $\{1, 2, \dots, 6\}$ and then rolls 3 dice.

The player wins \$1 for each die that matches the number, he loses \$1 if no die matches the number.

What is the expected outcome of that game:

Consider a game in which a player chooses a number in $\{1, 2, \dots, 6\}$ and then rolls 3 dice.

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What is the expected outcome of that game:

$$-1\left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)^3 = -\frac{17}{216}.$$

Linearity of Expectation

Theorem

For any two random variables X and Y

$$E[X + Y] = E[X] + E[Y].$$

Theorem

For any constant c and discrete random variable X ,

$$E[cX] = cE[X].$$

Note: X and Y do not have to be independent.

Examples:

- The expectation of the sum of n dice is . . .

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- The expectation of the outcome of one die plus twice the outcome of a second die is . . .

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- How many people we expect to have gotten their own coats?

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- $E[X_i] = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) =$
- $\Pr(X_i = 1) = \frac{1}{N}$
- $E[X] = E \left[\sum_{i=1}^N X_i \right] = \sum_{i=1}^N E[X_i] = 1$

Bernoulli Random Variable

A **Bernoulli** or an **indicator** random variable:

$$Y = \begin{cases} 1 & \text{if the experiment succeeds,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[Y] = p \cdot 1 + (1 - p) \cdot 0 = p = \mathbf{Pr}(Y = 1).$$

Binomial Random Variable

Assume that we repeat n independent Bernoulli trials that have probability p .

Examples:

- I flip n coins, $X_i = 1$, if the i th flip is “head” ($p = 1/2$)
- I roll n dice, $X_i = 1$, if the i th die roll is a 4 ($p = 1/6$)
- I choose n cards, $X_i = 1$, if the i th card is a J, Q, K ($p = 12/52$.)

Let $X = \sum_{i=1}^n X_i$.

X is a Binomial random variable.

Binomial Random Variable

Definition

A binomial random variable X with parameters n and p , denoted by $B(n, p)$, is defined by the following probability distribution on $j = 0, 1, 2, \dots, n$:

$$\Pr(X = j) = \binom{n}{j} p^j (1 - p)^{n-j}.$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways that we can select k elements out of n .

Expectation of a Binomial Random Variable

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Expectation of a Binomial R. V. - 2nd way

Using linearity of expectations

$$\mathbf{E}[X] = \mathbf{E} \left[\sum_{i=1}^n X_i \right] = \sum_{i=1}^n \mathbf{E}[X_i] = np.$$